SVM-6

Optimum Shock and Vibration Isolation

The Shock and Vibration Information Center
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Chapter 1

INTRODUCTION

This monograph deals with contemporary approaches to the problem of optimum shock and vibration isolation design. Isolation devices act to reduce the unwanted effects of shock and vibration disturbances on critical elements of a mechanical system. The problem of optimum design has to do with the selection of isolators that cause an index of the system performance to be optimized; that is, to take on a value either less or greater than that associated with other candidate isolators. In addition, the optimum design usually must satisfy constraints which are imposed on other aspects of the system response and the parameters which describe the isolators.

The essential elements of the optimum design problem, in addition to a description of the system kinematics and dynamics, are threefold; namely,

- The performance index
- The design constraints
- The shock or vibration excitation.

These aspects are considered in Chapters 2 and 3, followed by a general formulation of the optimum design problem in Chapter 4. Solution methods depend on the nature of the input excitation, i.e., whether deterministic or random and, for the former, whether a discrete shock pulse or a periodic vibration. Particular solution methods and results for these different excitation forms are covered in the remaining four chapters of the monograph. An annotated bibliography is included, as well as a glossary and an appendix dealing with a linear programming formulation for a class of shock isolation systems.

Though the notion of optimization is implicit in the design process, optimization as a formalized approach to engineering design is a relatively new concept. The engineer's function always has been to produce a final design that is better in some way than possible alternatives. However, it is in the selection of a quantitative measure of performance, i.e., an index whose numerical value serves to rank otherwise acceptable candidate designs, that an optimum design is distinguished from a conventional, or merely acceptable, design.

It is important to keep in mind that the sense of the optimization is totally dependent upon the choice of performance index and that, with respect to this index, what is termed the *optimum* is simply a system whose performance is better (or at least no worse) than that of the candidate designs with which it has been

compared. Thus, it cannot be said that some as-yet-unthought-of design is not *better* than the optimum; or that a better measure of optimization cannot be found.

While reliance on a performance index as the basis for the optimization seems unavoidable, limitations to the generality of the optimization, as a consequence of comparison with a limited class of alternatives, can be surmounted. The optimization techniques of modern control theory, when applied to synthesis in the time or frequency domain, make it possible to establish bounds on the performance index for all admissible candidate isolator concepts. In the time domain, this approach, termed *time-optimal synthesis,* amounts to describing the way in which an isolation system would have to respond for the performance index to take on the least possible value consistent with the constraints; however, it says nothing of what the isolator element ought to be from a device-oriented point of view to achieve this optimum performance. Still, this is valuable information to the designer; information he almost never otherwise possesses. While the literature on optimum control theory is extensive, there is extremely little with directed application to mechanical system design. Within these limits, we have attempted to be comprehensive in treating this newer approach to design optimization. Chapter 5 deals with applications to shock isolation systems. More limited results for vibration isolation under harmonic excitation are presented in Chapter 7, and for random excitation in Chapter 8.

After establishing optimum performance bounds, the designer is still faced with the problem of selecting specific isolator concepts to achieve the desired performance characteristics. We do not address this problem from a hardware point of view; that is simply not the focus of this monograph. Rather, it is assumed that candidate isolator concepts have been selected by some means and that there remains the problem of identifying the open design parameters (e.g., spring constants, damping rates) so that the resulting system performance is optimum. Here *optimum* refers to the performance achievable by the particular class of isolator elements under consideration. The difference between this optimum and the above-mentioned optimum performance bound represents the margin for improvement in system performance which theoretically can be achieved with an isolator concept. It is this ability to evaluate the extent to which an optimum design can be improved that the designer seldom, if ever, learns from conventional optimization methods, and we give it great emphasis in the monograph.

We term the problem of identifying the optimum set of parameters for an otherwise specified isolator concept that of *design-parameter synthesis.* Cast in discrete numerical terms, this is recognized as a problem in mathematical programming for which various methods of solution are available. The applicability of these methods generally depends on the analytical form of the system equations, performance index, and constraints. The more conventional approach, termed *direct synthesis,* employs numerical search techniques. We establish the problem formulation and point out the complications which result with increasing

dimension of the system, but do not deal directly with methods of solution. For specific solution techniques, the reader is referred to an extensive literature.

Another method, termed *indirect synthesis*, makes use of the results of the ideal isolator response characteristics determined in conjunction with the optimum performance bounds. This is a newer method, and, while the results reported so far appear very promising, the extent of its applicability remains to be established. However, in view of the potential computational advantages of this approach over the direct method, we have included it in the monograph. Both methods are presented in Chapter 6 for shock isolation systems; application to vibration isolation systems under harmonic and random excitation is discussed in Chapters 7 and 8, respectively. Chapter 6 also includes material on the sensitivity of optimum shock isolator designs to uncertainties in the input excitations.

System design includes system analysis as a subset. For shock and vibration isolation systems of any complexity, the analysis involves the solution of sets of differential equations, which for practical reasons must be approached by numerical means. Thus, our treatment of the design process is very definitely oriented toward computational methods requiring large digital computers. The limitations of the available literature for the most part necessitate the use of simple examples which either possess closed-form analytical solutions or are not particularly demanding on the computational methods. This is unfortunate, since the dimension of the computational problem usually determines the practicality of the method of solution. Despite the lack of convincing examples, however, we have tried to emphasize methods whose applicability extends to the larger, real-world systems.

Chapter 2

OPTIMIZATION CRITERIA

By *optimization criteria* we refer to both the performance index, which is the basis for ranking competing designs, and constraints, which serve to restrict the designs from which the optimum is selected. A simple example will illustrate the significance of these criteria. Figure 2.1 shows a single-degree-of-freedom (SDF) isolator system consisting of a parallel spring and damper interposed between the package (rigid mass) and the (massless) base structure. The base is subjected to a prescribed shock pulse. Let us assume that the rattlespace (i.e., the maximum displacement of the package relative to the base) is to be as small as possible, but under no circumstances can the peak acceleration experienced by the package exceed a certain amount. In this case, we would select rattlespace as the *performance index* and treat the peak transmitted acceleration as the *response constraint*. In comparing isolators with different spring rates and damping coefficients, only those that satisfy the acceleration bound would be considered acceptable designs, and among those the one requiring the least rattlespace would be the optimum.

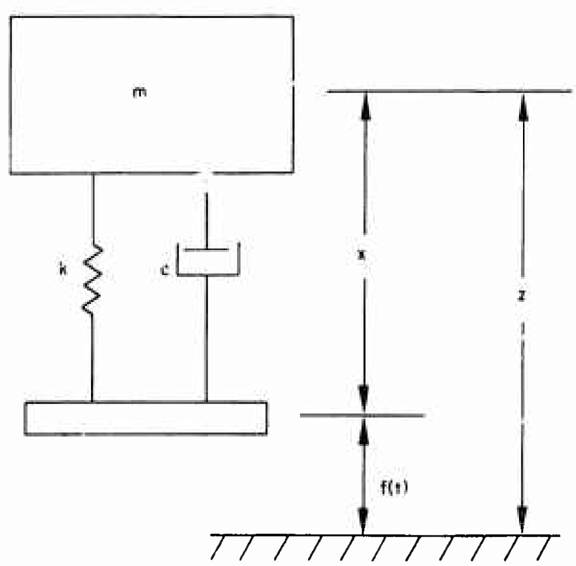


Fig. 2.1. An optimum isolator design problem.

This example points up the significance of the constraints. Were it not for the restriction placed on the acceleration of the package, the rattlespace could be reduced to zero by using an extremely stiff spring. Then, however, the package would experience the same peak acceleration as contained in the input shock. The restricted nature of this sort of optimum design problem also is suggested, since finding the best linear spring-dashpot isolator tells the designer nothing concerning the further reductions in rattlespace that might result from the use of a nonlinear spring. Also, the optimum choice of spring and damper might be considerably different for another shock input or for another choice of performance index.

While the specific choice of the optimization criteria depends on the design situation, it is useful to set forth some general functional forms that will be employed throughout the monograph.

2.1 Performance Index

Our primary assumption is that a single index of system performance can be selected as the basis for optimization. However, as will be discussed, this is not an overly restrictive assumption. The performance index, which we will denote by ψ, may be a function of the system response variables (as in the preceding example) or of the isolator design variables, or of both. The latter situation would be encountered when the performance index is related to such measures of system effectiveness as cost, maintainability, or reliability, all of which depend in a complicated way on the details of specific isolator concepts and configurations. Such a performance index poses no particular problems so far as the synthesis problem is concerned. However, the ability to establish theoretical limits to the performance index without prior choice of isolator configuration requires that the index be expressible only in terms of system response quantities. For this reason, we will consider only such forms for ψ.

Deterministic Forms

The optimum design problem for shock isolation systems (Chapters 5 and 6) and harmonic vibration isolation systems (Chapter 7) is presented from a deterministic point of view. Here the performance index is selected as the maximum absolute value of some response quantity with respect to time.† The response quantity is denoted by h, and the performance index then is

$$\psi = \max_t |h|. \tag{2.1}$$

†The notation and usage within this section agrees with that of Chapters 5 and 6, unless otherwise indicated. Although Chapter 7 also deals with a deterministic problem, time does not play the same role and there is an adjustment in the functional form for ψ.

The quantity h may be a displacement, velocity, acceleration, stress, or some combination of these, and is found from the solution of the system equations of motion.

An extended form of Eq. (2.1) results when ψ is defined to be the largest of several related peak response quantities. For example, it may be desired to minimize the largest of the stresses occurring at three locations in the system. If we denote stresses at these points by σ_1, σ_2, and σ_3, then we may write the performance index as

$$\psi = \max\left[\max_t|\sigma_1|, \max_t|\sigma_2|, \max_t|\sigma_3|\right].$$

Therefore, a more general form of Eq. (2.1) is

$$\psi = \max_s \max_t |h_s|; \qquad s = 1, 2, \ldots, S, \tag{2.2}$$

where we select the largest among S response quantities, h_s, for the performance index. This form also is applicable when the position of the maximum value of h is unknown in advance.

Equation (2.2), of course, requires that the quantities h_s be of comparable type. Reference 1 considers a performance index involving different response quantities. With reference to the example system of Fig. 2.1, the index is

$$\psi = \max_t|\ddot{z}| + \rho \max_t|x|. \tag{2.3}$$

Here, $\ddot{z}$ is the acceleration of the package mass, x the relative displacement, and the constant ρ a weighting factor. For large values of ρ, the performance index favors rattlespace, while for small values of ρ, the preference is for peak acceleration.

Another form of performance index is the integral of a response quantity over some time period of interest. Thus,

$$\psi = \int_{t_1}^{t_2} H(h)dt, \tag{2.4}$$

where H is a known function, e.g., $H(h) = h^2$. This type of performance index is common in control theory applications and is related to the forms considered by classical analytical optimization techniques (i.e., calculus of variations). While the solution techniques to be presented are equally applicable to the integral performance index, we do not specifically deal with this form, as physical motivation seems lacking for isolation system applications. It might be noted that the

integral form reduces to the maximum value form in the following limiting situation [2]:

$$\psi = \lim_{p \to \infty} \left[\int_{t_1}^{t_2} |h|^p dt \right]^{1/p} . \tag{2.5}$$

As Ref. 3 shows, even small values of p can yield results close to those found with the $\max_t |h(t)|$ criterion.

It is possible to introduce a supplementary performance index at a time when the primary index has been satisfied, since, subsequently, the response may not be unique. For example, in the single degree-of-freedom (SDF) example system, the isolator force trajectory which minimizes ψ may have any value at sufficiently late times for which min ψ is not exceeded. To identify a particular isolator force trajectory, it may be required to bring the mass to rest in minimum time, this latter condition serving as the supplementary performance index. This approach is discussed in Ref. 4 and 5. The relative merits of various performance indices are studied in Ref. 1.

Statistical Forms

Statistical forms of the performance index are considered in Chapter 8 with application to random vibration isolation. Each of the deterministic forms has its probabilistic counterpart. For example, the response quantity h can be considered as the expected value of an appropriate random response variable or combination of such variables, and a performance index such as Eq. (2.1) is then meaningful. Also, the performance can be based on the probability of not exceeding the maximum value of a response function, as in Ref. 6.

Reference 1 considers a form based on expected mean square values which is somewhat analogous to Eq. (2.3). Here

$$\psi = \mathbf{E}[\ddot{z}^2] + \rho \mathbf{E}[x^2],$$

where the $\mathbf{E}[\ \ ]$ refer to expected mean square responses for the SDF system of Fig. 2.1, the input being a random vibration. As before, ρ is a constant weighting factor. A related form of the performance index is considered in Ref. 7 as

$$\psi = \ddot{z}_0^2 + \rho x_0^2.$$

Here, $\ddot{z}_0$ and x_0 are values of the random variables $\ddot{z}$ and x such that the probability of $|\ddot{z}| < z_0$ and $|x| < x_0$ is equal to a prescribed level.

2.2 Constraints

The constraint functions serve to restrict the system response or limit the choice of isolator parameters as required by design considerations. We make a distinction between two types of constraints: (a) response constraints, which involve the system response variables, and (b) parameter constraints, which involve the design parameters associated with particular isolator element configurations. Both types of constraint functions will be denoted by the symbol C_k, the subscript indicating that there are $k = 1, 2, \ldots, K$ of these.

Response Constraints

Response constraints are limitations enforced on such physical response quantities as stress, displacement, and acceleration. In the SDF example discussed, the single constraint was imposed on the peak accelerations of the package. If we denote the maximum allowable acceleration by A, then this constraint may be written

$$\max_t |\ddot{z}| \leqslant A,$$

or equivalently,

$$-A \leqslant \ddot{z} \leqslant A.$$

In all instances, the response constraints that we consider may be written as two-sided inequalities, as in this example. Therefore, the general expression for the K_1 response constraints is

$$C_k^L \leqslant C_k \leqslant C_k^U; \qquad k = 1, 2, \ldots, K_1. \tag{2.6}$$

For deterministic systems, the bounding values C_k^L, C_k^U may be constants or functions of time. In the preceding example, $K_1 = 1$, $C_1 = \ddot{z}$, and $C_1^L = -A$, $C_1^U = A$. For random vibration isolation, the response constraints may be in terms of expected values for which Eq. (2.6) still applies. Or, the constraints could require the probability of some response quantity exceeding a specified level to be less than a fixed value.

Parameter Constraints

Parameter constraints refer to limitations on the design parameters describing a particular isolator concept or configuration. In the linear spring-dashpot isolator element (Fig. 2.1), we would require both the spring rate k and damping coefficient c to be positive numbers. This could be expressed as

$$k \geqslant 0; \quad c \geqslant 0.$$

Strict inequality forms would be used if the possibility of omitting either the spring or the damper from consideration were to be avoided. If, in addition, it was desired that the isolator element be overdamped, then the constraint $c > 2\sqrt{km}$ would be added.

The parameter constraints usually will be expressed as one-sided inequalities of the form

$$C_k \geqslant 0; \quad k = 1, 2, \ldots, K_2. \tag{2.7}$$

However, two-sided inequalities may occur, as, for example, when a spring rate is required to lie between prescribed values. Also included is the situation where a design parameter must assume only discrete prescribed values.

Chapter 3

SHOCK AND VIBRATION ENVIRONMENTS

Mechanical systems must be designed to function adequately in a wide range of dynamic environments. These environments are normally classified as shock or vibration, depending on the duration of the disturbance. Shock is said to occur when the system is acted upon by a "sharp," aperiodic disturbance lasting a relatively short period of time. Vibration, in contrast, is characterized by an oscillatory disturbance extending over a relatively long period of time.

The sources of shock and vibration environments are numerous and difficult to categorize [8]. We make no attempt to do so here, but rather concern ourselves with the mathematical representations of these environments which are necessary to formulate the optimum design problem. The assignment of specific numerical waveforms or equivalent parametric representations as related to a particular service environment is beyond the scope of the monograph.

Our characterization of shock environment will be entirely deterministic, whereas both deterministic and statistical forms are considered for vibration environments. These environments, or input disturbances, may be expressed in terms of force, displacement, velocity, or acceleration. Even for deterministic forms, the design problem is formulated so as to allow uncertainty in the inputs.

3.1 Shock Environment

Input Waveform Description

In the simplest case, it is assumed that the shock pulse, denoted by $f(t)$, is a known function of time over a prescribed interval, $t_0 \leqslant t \leqslant t_f$. This may be described analytically or in discrete digital form. A number of such input waveforms may be specified at different points in the system, in which case their relative time phasing is also known.

A more complicated, but more real-world, situation is where the possibility of different waveforms must be considered. We term this a multiple-input specification and assume that any one of a finite number of prescribed waveforms $f_\ell(t)$ ($\ell = 1, 2, \ldots, L$) is equally likely to occur at each input point of the system.

Figure 3.1a shows a family of three ($L = 3$) acceleration pulses that might be prescribed as shock inputs. Multiple inputs represent the usual situation in design, optimum or otherwise, and require that all permutations be considered. The problem is not quite so direct, however, in the determination of optimum performance bounds, as is discussed in Chapter 5.

Input Class Description

Another means of introducing uncertainty into the specification of the shock environment while maintaining a deterministic representation is to describe a class of inputs that contains an infinite number of waveforms, any of which is equally likely to occur. The class may be described by a time-varying band about a nominal acceleration waveform, as suggested in Fig. 3.1b, or bounds unrelated to waveform may be prescribed (Fig. 3.1c). An additional requirement may be imposed as, for example, that some function of the waveform averaged over time be given or bounded (Fig. 3.1c). In fact, a fixed set of rules for describing a class need not be stated, and no particular concern need be held for a convenient mathematical description of the class. This is a fairly natural way of describing the environment, as it recognizes what is known and what is not. Of course, special solution techniques are required if the number of individual waveforms is unlimited.

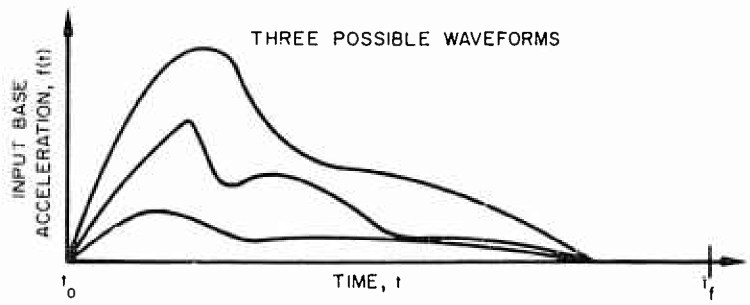
Input Shock Spectra

The shock spectrum, which portrays the maximum response (i.e., displacement, velocity, and acceleration) of an SDF linear-mass-spring system (usually undamped) to the input waveform over a range of frequencies, conventionally is utilized as a characterization of the shock input. Since this information retains nothing of the time details of the input, and it is not possible to infer these from the spectral plot, we do not consider this form of input representation.

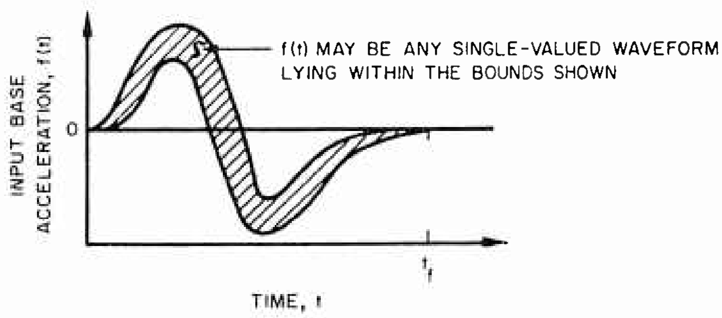
3.2 Vibration Environment

Harmonic Inputs

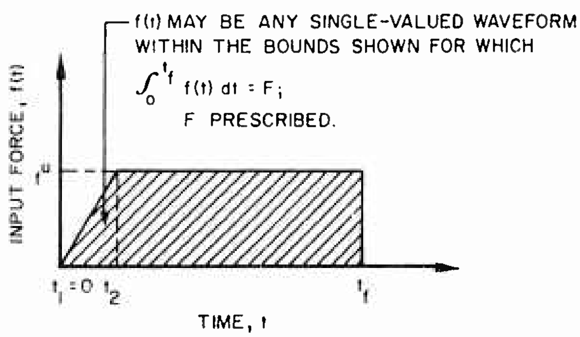
Harmonic vibration environments generally are represented as time-varying quantities in the form of Fourier series. The solution methods and results presented in Chapter 7 are limited to simple harmonic inputs; i.e., waveforms of either sine or cosine form, e.g., $f(t) = f_m \sin \omega t$. It is possible to represent uncertainties of harmonic waveforms in terms of frequency and amplitude (ω vs f_m) bounds (Fig. 3.2) and otherwise parallel the preceding description of shock pulses. Reference 9 is suggested for a general discussion of periodic waveforms.



(a) Multiple input



(b) Input class



(c) Input class

Fig. 3.1. Descriptions of shock pulses.

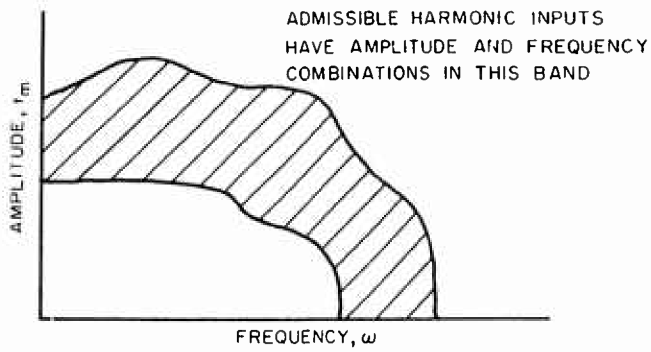


Fig. 3.2. Frequency-amplitude spectrum for harmonic disturbances.

Random Inputs

By random inputs we refer to aperiodic waveforms whose magnitudes are random variables. A narrow-band random disturbance resembles a harmonic input in that it possesses a principal frequency component, but differs in that its magnitude varies randomly. In contrast, no dominant frequency component can be identified for a wide-band random disturbance. A spectral representation of the random input is sufficient for the optimum design of linear isolator systems as presented in Chapter 8. Hence, we will review this concept briefly.

We denote the random input by $f(t)$ and assume that it is a stationary random function of time t. The autocorrelation function $R(\tau)$ is defined to be

$$R(\tau) = \mathbf{E}[f(t)f(t+\tau)], \tag{3.1}$$

where $\mathbf{E}[\ ]$ denotes expected value, or ensemble average. For the class of $f(t)$ under consideration, $R(\tau)$ is computed as

$$R(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)f(t+\tau)d\tau. \tag{3.2}$$

Then the spectral density S is defined as

$$S(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\tau)\exp(-i\lambda\tau)d\tau. \tag{3.3}$$

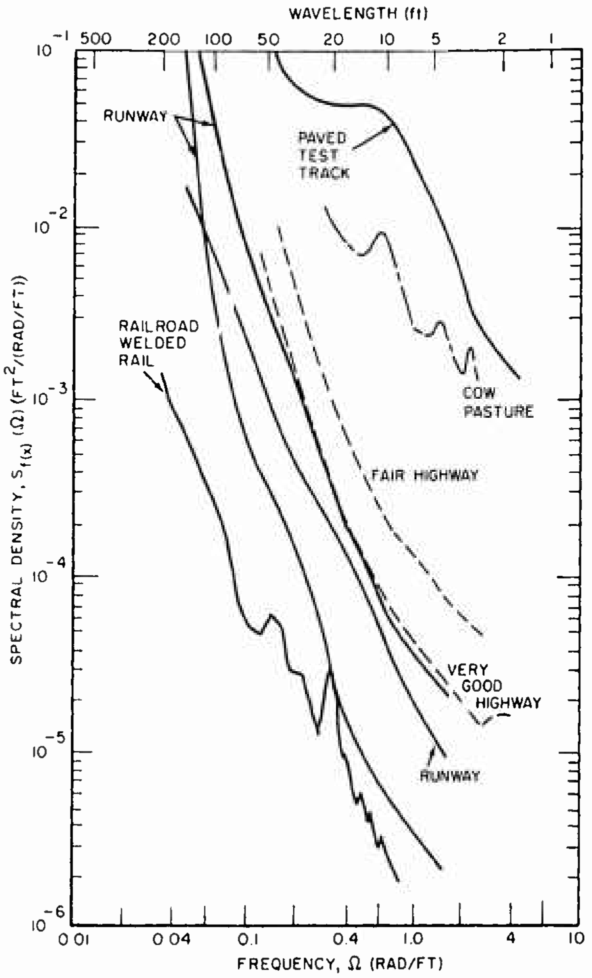


Fig. 3.3. Spectral densities of various terrains.

Observe that $S(\lambda)$ is the Fourier transform of the autocorrelation function. The transform parameter λ is identified with the frequency content of the disturbance.

An application of these concepts is considered in Chapter 8 with respect to the measured profiles of roadways and track systems which constitute the vibration input to ground vehicles. The height of the road or track above some datum is denoted by f and assumed to be a stationary random function of distance x. Then the autocorrelation function and spectral density can be evaluated in spatial rather than temporal terms in exactly the above manner. In place of the time frequency λ, there is the spatial frequency Ω in units of radians per length. If a vehicle moves across the profile with constant speed V, the relationships $x = Vt$ and $\lambda = V\Omega$ hold, and

$$VS_{f(t)}(\lambda) = S_{f(x)}(\Omega). \tag{3.4}$$

Here we have used subscripts to distinguish between the time and space spectral densities. Equation (3.3) is used to compute $S_{f(t)}(\lambda)$, and the expression for $S_{f(x)}(\Omega)$ is

$$S_{f(x)}(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\xi)\exp(-i\Omega\xi)d\xi. \tag{3.5}$$

On the basis of measured spectra for highways, runways, test tracks, and even a cow pasture [10-13], it is suggested that $S(\Omega)$ can be approximated over a wide range of frequencies by

$$S_{f(x)}(\Omega) = c_1\Omega^{-c_2},$$

where c_1 and c_2 are positive constants. Representative measurements are summarized in Fig. 3.3, from which we observe that $c_2 \approx 2$. If $S_{f(x)}(\Omega) = c_1\Omega^{-2}$, then the spectral density for the profile slopes $f'(x)$ will be constant, as will $S_{f(t)}(\lambda)$ evaluated for the velocities $\dot{f}(t)$. A random process whose spectral density is constant is said to be white noise, which implies a wide-band disturbance. To the extent that the design optimization procedure can be carried out for a spectral characterization of the profile, this means that the input involves only a single independent parameter. Reference 14 deals at length with this model of the environment.

Chapter 4

OPTIMUM ISOLATOR DESIGN FORMULATION

By combining the optimization criteria and the shock and vibration environment information with the system dynamics, we can develop a general problem statement for optimum isolator design. We strive to emphasize the generality of the formulation despite the fact that, for the most part, only rather simple systems have been solved to date. Two formulations of the general synthesis problem are presented. One, *optimum design-parameter synthesis,* selects the optimum isolator from among a preselected class of isolators as well as the minimum performance index for this class; the other, termed *time-optimal synthesis,* establishes the absolute minimum for the performance index but does not describe the optimum isolator in hardware-oriented terms. Finally, we consider a description of the optimum performance characteristics applicable to either problem formulation. Methods of solution are dealt with in subsequent chapters.

4.1 Optimum Design-Parameter Synthesis

General Isolation System

In hardware terms, an isolator is a device interposed between elements of a structural dynamic system† to reduce to tolerable levels transmitted effects of the external shock or vibration environment for designated system elements. Among isolators that acceptably achieve this function, the optimum isolator is the one that causes an index of the system performance to take on its minimum value. By design-parameter synthesis we refer to the selection of the optimum isolator from among a preselected class of candidate isolators that differ only in the numerical values of certain open parameters.

The design synthesis process requires that a mathematical model for the physical system, including the candidate isolator devices, be postulated. This is the starting point for our consideration. In the most general case we assume that this model can be described qualitatively as a multiple-isolator, multiple-degree-of-freedom (MDF) system and, quantitatively, by a system of nonlinear ordinary or partial differential equations.

†No limitation in generality is implied; the dynamic system may be composed of structural, mechanical, hydraulic elements, etc., or combinations of these.

Such a general dynamical system is shown schematically in Fig. 4.1. Two types of elements are considered; structural elements and isolator elements. There may be any number of each; in particular, we admit the possibility that M structural elements and J isolator elements are interconnected in an arbitrary fashion. A structural element may represent a discrete mass point, a rigid body of distributed mass, or a flexible structure such as a framework or a shell. The isolator elements, similarly, can represent either simple mechanisms without mass or models of more complicated devices. In general, the structural elements constitute the prescribed portions of the system (i.e., the base structure and the elements to be isolated) and the isolator elements are to be chosen in accordance with the design objectives.

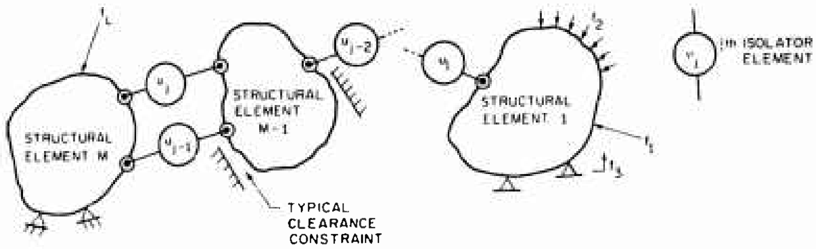


Fig. 4.1. Multiple-degree-of-freedom system.

The notation used is as follows:

$\mathbf{z}_m$; $m = 1, 2, \ldots, M$ are the position vectors that define the initial configuration of the mth structural element in some convenient local coordinate system.

$\mathbf{w}_n(\mathbf{z}_m, t)$; $n = 1, 2, \ldots, N$ are the generalized coordinates, i.e., any kinematically acceptable choice of position vectors that define the state of the mth structural element over the entire time range of interest. Usually these will define points on each element relative to its initial configuration.

$\mathbf{u}_j$; $j = 1, 2, \ldots, J$ are the forces (moments) in the jth isolator element. In general, each force (moment) is a three-component vector. The functional dependence of the $\mathbf{u}_j$ will be considered later.

$\mathbf{f}_\ell(\mathbf{z}_m, t)$; $\ell = 1, 2, \ldots, L$ are the input disturbances applied at various positions in the system. These may be in the form of distributed or concentrated forces (moments) or prescribed motions of the system supports. Later this notation will be expanded to distinguish between several input positions and the possible occurrence of more than one input a given position.

The positions of the mass particles composing the mth structural element is governed by a set of differential equations of the form

$$\mathcal{L}_n[\mathrm{w}_n(\mathrm{z}_m, t)] = \mathrm{G}_n(\mathrm{u}_j, \mathbf{f}_\ell); \quad n = 1, 2, \ldots, \mathrm{N} \tag{4.1}$$

$$\left.\begin{aligned} \mathrm{w}_n(\mathrm{z}_m, t_0) &= \mathrm{w}_n^0 \\ \frac{\partial \mathrm{w}_n}{\partial t}(\mathrm{z}_m, t_0) &= \dot{\mathrm{w}}_n^0. \end{aligned}\right\} \tag{4.2}$$

Here, $\mathcal{L}_n$ represents a second-order differential operator (ordinary or partial) and G_n is a function containing only the isolator forces and the input disturbances. Equation (4.2) represents the initial conditions. An appropriate set of boundary conditions also must be prescribed,

The mathematical description of the isolator forces has an important bearing on the problems that can be solved. From the design point of view we limit our considerations to so-called passive isolators; i.e., isolators for which the force expression u_j depends explicitly on the change in an associated displacement, velocity, or both. Assume that the jth isolator connects the m and m-1 structural elements, and that the position vectors of the attachment points are denoted by $\mathrm{w}_n(\mathrm{z}_m, t)$ and $\mathrm{w}_n(\mathrm{z}_{m-1}, t)$. We define the relative displacement of the jth isolator to be

$$x_j(t) = q\,[\mathrm{w}_n(\mathrm{z}_m, t), \quad \mathrm{w}_n(\mathrm{z}_{m-1}, t)], \tag{4.3}$$

where q is the appropriate kinematic function. A passive isolator is one for which the force magnitude can be expressed in the form

$$u_j = u_j(x_j, \dot{x}_j). \tag{4.4}$$

Design-parameter synthesis involves selecting an appropriate set of parameters that describe the isolator element in question. We will consider that there are R_j such parameters associated with the jth isolator and denote them by a_{jr}. Then, a more explicit form of Eq. (4.4) is

$$u_j = u_j(x_j, \dot{x}_j, a_{jr}); \qquad r = 1, 2, \ldots, R_j. \tag{4.5}$$

The difference in notation is best emphasized by example. If the isolator $j = 1$ consists of a linear spring-dashpot element, as shown in Fig. 2.1, the simplest expression for the force is $u_1 = kx_1 + c\dot{x}_1$, where k is the spring rate and c the damping coefficient. If k and c are known values, then we use Eq. (4.4) with

$$u_1 = u(x_1, \dot{x}_1) = kx_1 + c\dot{x}_1.$$

If, however, our problem is to select optimum values of k and c, these would be unknowns in the design problem and we would use Eq. (4.5) with

$$R_1 = 2; \quad a_{11} = k \text{ and } a_{12} = c.$$

The discussion of the performance index and constraints in Chapter 2 did not refer to a functional representation for these quantities. We are now in a position to do so. It is usually convenient to indicate an explicit dependence on the force function u_j since, in one form or another, these are the problem unknowns. Thus, in general the performance index, Eq. (2.2), is written

$$\psi = \max_{s} \max_{t} |h_s(t, \mathrm{u}_j)|; \quad s = 1, 2, \ldots, S, \tag{4.6}$$

where, it will be recalled, the largest of S comparable response quantities, h_s, is selected to be the index.

The response constraints, Eq. (2.6), are written as

$$C_k^L(t) \leqslant C_k(t, \mathrm{u}_j) \leqslant C_k^U; \quad k = 1, 2, \ldots, K_1. \tag{4.7}$$

The possibility of constant bounds to the constraint function, i.e., C_k^L, C_k^U = constant, is included as a special case.

Parameter constraints, Eq. (2.7), can be written as

$$C_k(a_{jr}) \geqslant 0; \quad k = 1, \ldots, K_2 \tag{4.8}$$

or

$$C_k^L \leqslant C_k(a_{jr}) \leqslant C_k^U; \quad k = 1, \ldots, K_3. \tag{4.9}$$

The mathematical statement of the optimum isolation design problem can now be stated as follows. We are given a dynamical system comprising M structural elements and J isolator elements. Coordinate systems are defined, in terms of which all of the isolator force functions, Eq. (4.5), are known. However, the numerical values of the design parameters a_{jr} remain to be determined; there are R_j of these for each of the J isolators. The input disturbances are known as to position of application, waveform, and relative time phasing. Thus, the equations of motion (Eq. (4.1)), initial conditions (Eq. (4.2)), and appropriate boundary conditions are known.†

†This problem statement includes the case of multiple inputs to the extent that Eq. (4.1) must be solved repetitively for each waveform.

Comparable functions of the system response, h_s, are selected as the basis for the performance index. These are to be evaluated from the solution to the equations of motion. If the maximum value of $h_1, h_2, \ldots, h_S$ over the time range of interest is established, then the largest of these maxima is identified as the performance index ψ in accordance with Eq. (4.6).

A number K_1 of constraint functions are prescribed in terms of the response quantities, Eq. (4.7), and other constraints involving the unknown design parameters, Eq. (4.8) and (4.9), also are prescribed; there are $K_2 + K_3$ of the latter.

The optimum design synthesis problem requires that we find the design parameters a_{jr} such that ψ is minimized and all constraint functions are satisfied. We emphasize again that by referring to this as the problem of design-parameter synthesis, we imply a preselection of isolator devices and are seeking to identify the best choice among the admissible design parameters. This formulation is expanded in Chapters 5 and 6 to include the situation where the inputs are of a class description comprising an infinite number of waveforms. Example 1 shows how this formulation may be applied.

4.2 Time-Optimal Synthesis

General Isolation System

An alternate formulation of the optimum design problem leads to a lower bound value of the performance index for any type of isolator consistent with the constraints. This differs from the previous design parameter synthesis formulation in that no a priori assumption is made regarding the functional form of the isolator force in terms of the relative state variables; i.e., the form of Eq. (4.4) or (4.5) is unknown. Instead, we consider the isolator force to be an explicit function of time $u_j(t)$, which we will synthesize in the time domain (for each value of j) so as to minimize the performance index and satisfy the constraints. We term this process *time-optimal synthesis.* The resulting value of the performance index is the best that can be accomplished for any isolator regardless of the hardware device utilized. The description of the isolator is provided by the optimum $u_j(t)$. This is not the usual device-oriented description, but rather a description of the manner in which the optimum device responds.

The problem formulation for time-optimal synthesis follows exactly as before except that the unknowns, instead of being the design parameters a_{jr}, are now the isolator force functions $u_j(t)$. Of course, the solution techniques are entirely different.

Quasi-Linear Isolation Systems

It is plausible to suppose that the solution methods are simplified when the governing equations are linear, and indeed that is the case. Linearity requires that the equations of motion, Eq. (4.1), be linear in the dependent variables.

When viewed as the problem of design-parameter synthesis (Section 4.1) this in turn requires that the isolator force functions, Eq. (4.4), depend linearly on the relative state variables. This is a severe limitation on the type of isolator device that may be considered and, hence, linear systems, when they imply linear isolators, are of limited practical value. From the standpoint of time-optimal synthesis, however, linearity merely requires that the equations of motion involve u_j linearly but does not restrict the form of Eq. (4.4). In other words, the isolators need not exhibit linear force (moment)-displacement or -velocity characteristics, although in all other respects the system is linear. Such systems are termed *quasi-linear.*

In summary, then, a quasi-linear system is one for which, in Eq. (4.1), the $\mathcal{L}_n$ are linear differential operators and the $\mathbf{G}_n$ are linear functions of the $\mathbf{u}_j$ and $\mathbf{f}_\ell$. The system considered in Example 1 is linear in all respects and, of course, is quasi-linear as well. However, if the linear spring-dashpot isolator were replaced by a nonlinear device, the system would still be quasi-linear.

The condition of quasi-linearity permits application of superposition to the solution to the equations of motion. If, in addition, the response functions $h_s(t, \mathbf{u}_j)$ and response constraints $C_k(t, \mathbf{u}_j)$ involve the u_j linearly, superposition may be used to construct these quantities as well. Thus, in general, we may write

$$h_s(t, \mathbf{u}_j) = h_{s0}(t) + \sum_{j=1}^{J} \int_0^t \mathbf{R}_{sj}(t-\tau) \cdot \mathbf{u}_j(\tau) dt$$

and (4.10)

$$C_k(t, \mathbf{u}_j) = C_{k0}(t) + \sum_{j=1}^{J} \int_0^t \mathbf{R}_{kj}(t-\tau) \cdot \mathbf{u}_j(\tau) d\tau.$$

Here, $\mathbf{R}_{sj}$ and $\mathbf{R}_{kj}$ are the appropriate responses to a unit force (moment) input at the attachment points of the jth isolator element. The vector notation serves to emphasize that, generally, a component of $\mathbf{R}$ exists for each component of $\mathbf{u}_j$, and the $\mathbf{R} \cdot \mathbf{u}$ notation signifies a dot-product summation over the respective components of each vector. The terms $h_{s0}(t)$ and $C_{k0}(t)$ are the responses to the L inputs $f_\ell(t)$. These contain the appropriate homogeneous solutions should the problem be stated with nonzero initial conditions. In the case of multiple inputs, $h_{s0}(t)$ and $C_{k0}(t)$ are separate functions for each of the prescribed combinations of waveforms.

We illustrate time-optimal synthesis in Example 2. The system is the same roadway vehicle considered in Example 1, except that here the isolator is of unspecified configuration.

4.3 Optimum Performance Characteristics

Single Input Waveform

Either design formulation, i.e., design-parameter synthesis or time-optimal synthesis, yields essentially two types of information: (a) the description of the optimum isolator and (b) the least value of the performance index consistent with the constraints. In design-parameter synthesis, the optimum isolators are described by the sets of parameters a_{jr}^* (asterisks are used to indicate optimum values), whereas in the time-optimal synthesis they are described by their force trajectories $u_j^*(t)$. In the first instance, ψ^* is a minimum only with respect to the preselected candidate isolators under consideration, and there may be other isolators for which ψ^* is smaller; in the second, ψ^* is an absolute minimum and no isolator can be found for which the system performance index takes on a lesser value. Thus, either formulation provides useful information, the more so whether a hardware-oriented description of the isolator or a lower bound to the performance index is of most interest. Here, we focus on the latter consideration.

Each value of ψ^* is associated with a prescribed input disturbance and fixed constraints. If the numerical values of the constraints are changed and the solution is repeated for the same input, another value of ψ^* is determined. In this manner, a relationship (i.e., sequence of values) between ψ^* and the constraints may be established. This relationship is termed the *optimum performance characteristic* of the system under consideration and is symbolically represented by $\psi^*(C_k)$. Generally, for K constraints, $\psi^*(C_k)$ is a hypersurface of $K + 1$ dimension; only in the case of one constraint, where $\psi^*(C_k)$ is a plane curve, can this function be portrayed by simple graphic means.

Consider an example of an SDF system where the performance index is selected as the displacement of the mass relative to its base and the absolute acceleration of the mass is constrained. This is illustrated in Fig. 4.2 for the design-parameter synthesis problem and in Fig. 4.3 for the time-optimal synthesis problem. Thus, we seek to minimize

$$\psi = \max_t |x|$$

for

$$\max_t |\ddot{z}| \leqslant A$$

and some prescribed $f(t)$. The curves in Fig. 4.4 suggest the relationship that might be found between ψ^* and A for various levels of the acceleration constraint A. The two curves are identified as the time-optimal and design-parameter solutions; the former provides the smaller values of ψ^* for the same A.

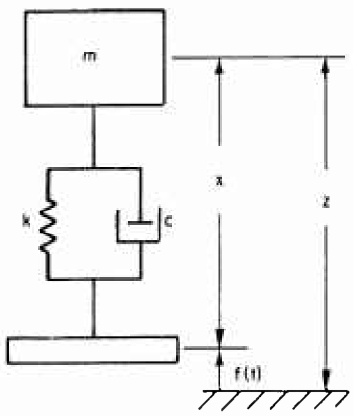


Fig. 4.2. An SDF system for design-parameter synthesis formulation.

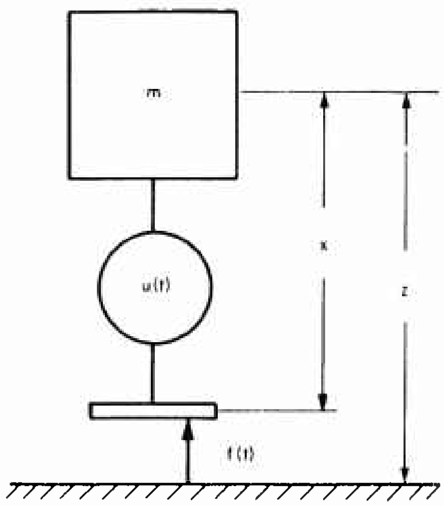


Fig. 4.3. An SDF system for time-optimal synthesis formulation.

Consider a point on the time-optimal solution, say Point 1 in Fig. 4.4, corresponding to the constraint level A_1. For this constraint level, ψ_1^* is the smallest value of the rattlespace that can be achieved with any isolator. Also, ψ_2^*, corresponding to Point 2 on the design-parameter solution, is the smallest rattlespace that can be achieved with a linear spring-dashpot isolator. The difference $\psi_2^* - \psi_1^*$ represents the improvement in performance over this linear isolator which theoretically is possible; of course, this result alone says nothing of what the ideal optimum isolator should be.

Each point along either of the two curves corresponds to a different isolator. That is, every point lying on the design-parameter solution corresponds to a different pair of parameter values (k^*, c^*), whereas every point on the time-optimal solution implies a different $u^*(t)$. For example, the intercept of the

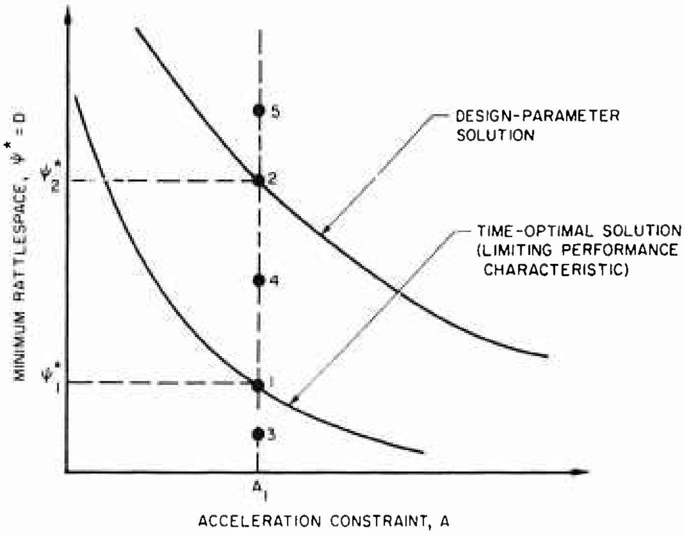


Fig. 4.4. Optimum performance characteristics.

time-optimal solution corresponds to the limiting cases of no isolator† ($A = 0$, $\psi^* = \max_t |f(t)|$), or a rigid isolator ($A = \max_t |\ddot{f}(t)|$, $\psi^* = 0$).

Each choice of isolator type leads to a different optimum performance characteristic, whereas a unique solution exists to the time-optimal solution. For this reason, we term the latter solution the *limiting performance characteristic* to distinguish it from the design-parameter solution.

A somewhat different application of the optimum performance data is provided by the design situation in which we want an isolator that yields a maximum acceleration A and requires the rattlespace D. The combination of values (A, D) is termed a design point and can be plotted in relationship to the optimum performance characteristics. For example, if the design point corresponds to Point 3 in Fig. 4.4, it is immediately recognized that no isolator can be found to meet these requirements, since the region below the limiting performance characteristic is not physically attainable. Without this information, the designer would soon discover the inadequacy of a linear spring-dashpot isolator but might never appreciate the hopelessness of his search. If the design point were to correspond to Point 4 in the figure, he knows that the linear isolator is inadequate but that conceivably some other isolator can be found to perform as desired. Point 5 corresponds to specifications which can be improved upon even by an optimum linear isolator.

There is reciprocity in the optimum performance characteristic relating to an interchange between the performance index and a constraint. Thus, the curves in Fig. 4.4 were described as the least rattlespace obtainable for a given

†Or a constant-force isolator, $u = mg$.

acceleration constraint. In each instance the same optimum performance characteristic would result if the problem were posed as that of finding the least possible maximum acceleration consistent with a prescribed rattlespace constraint [15]. It seems plausible that this should be the case for a multiple-constraint problem when the performance index is interchanged with any one of the response constraints.

Multiple Inputs

If the input disturbance is prescribed as a finite numbered family of waveforms, optimum performance characteristics can be obtained separately for each of the inputs (Fig. 4.4). The lower and upper bounds to these curves (for either formulation) represent, respectively, the most and least favorable performance characteristics associated with the family of inputs. Correspondingly, the particular waveforms that produce these bounding curves constitute "best" and "worst" disturbances among the prescribed family. The applications of these results depend on the circumstances of the problem and, in any event, are similar to the applications described for the single-input characteristic. The solution for the time-optimal formulation described in Chapter 6 is somewhat more involved than suggested here.

Input Class

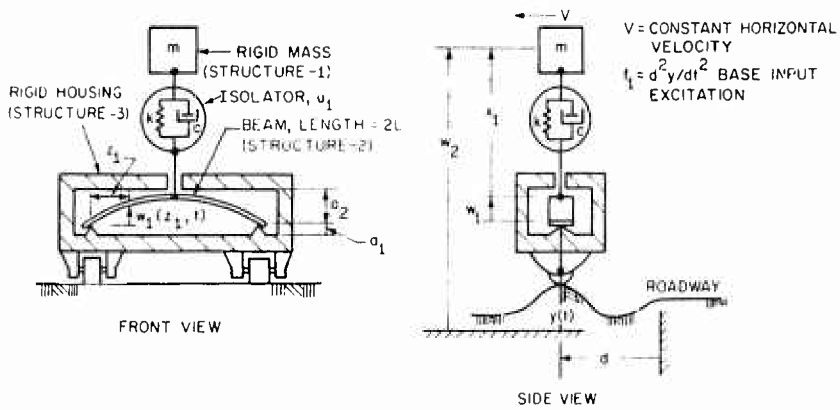
When the input disturbance is described as a class comprising an infinite number of equally probable waveforms, the concept of a best and worst disturbance still pertains, but these bounds cannot be found by enumeration. The solution to this problem is discussed in Chapter 6. Bounds to the optimum performance characteristics associated with these extreme disturbances are established as for the multiple input case.

Example 1

GENERAL ISOLATION SYSTEM PROBLEM: DESIGN-PARAMETER SYNTHESIS

Isolation System

We will consider the isolation system relating to a ground vehicle which encounters a discrete bump in an otherwise perfectly smooth roadway. The system involves three structural elements, a single isolator element, and one input. The primary vehicle structure, including wheel and axle, is modeled as a rigid housing. The vehicle moves with constant horizontal velocity V and, upon encountering the bump, follows its profile y exactly. The package to be isolated is represented by a rigid mass m and is connected through an isolator element u_1 to the center of a flexible beam located within the vehicle. The isolator is a linear spring-dashpot device (see figure) modeled as being massless, i.e., $u_1 = kx_1 + c\dot{x}_1$.



We want to select the spring rate k and damping coefficient c so that the peak acceleration experienced by the package is minimized. We also require that neither the beam nor the package deflect so far as to contact any portion of the surrounding vehicle structure. Finally, we limit consideration to values of k between k^L and k^U and require that the system be overdamped. The performance index and constraints are written in the following manner.

Performance Index

The performance index is the maximum acceleration of the package mass. Thus, the acceleration $\ddot{w}_2$ is the single response function of interest; $h_s = h_1$ and $S = 1$. According to Eq. (4.6),

$$\psi = \max_t |\ddot{w}_2|.$$

Response Constraints

There are two response constraints ($K_1 = 2$), one, say, C_1, which avoids bottoming of the beam, and the other, C_2, which avoids bottoming of the package. Thus, C_1 is a condition on w_1, and C_2 may be prescribed in terms of the relative displacement of the isolator terminals x_1. Thus, with reference to Eq. (4.7),

$$C_1(t) = w_1(L, t)$$

with

$$C_1^L = a_1; \quad C_1^U = a_2$$

and

$$C_2(t) = x_1(t) = w_2(t) - w_1(L, t) - y(t)$$

with

$$C_2^L = a_3; \quad C_2^U = a_4,$$

where a_3 and a_4 are prescribed dimensions (not shown in the figure).

Parameter Constraints

There are two parameter constraints, one requiring the spring rate to be within prescribed values ($K_3 = 1$, Eq. (4.9)) and the other ensuring that the package mass is overdamped ($K_2 = 1$, Eq. (4.8)). These may be written

$$C_3(k, c) = c - 2\sqrt{km} \geqslant 0$$

and

$$C_4(k) = k,$$

with

$$C_4^L = k^L; \quad C_4^U = k^U.$$

We must now express the state variables w_1 and w_2 and the acceleration $\ddot{w}_2$ as explicit functions of time and the two design parameters k and c. In the notation of Eq. (4.5),

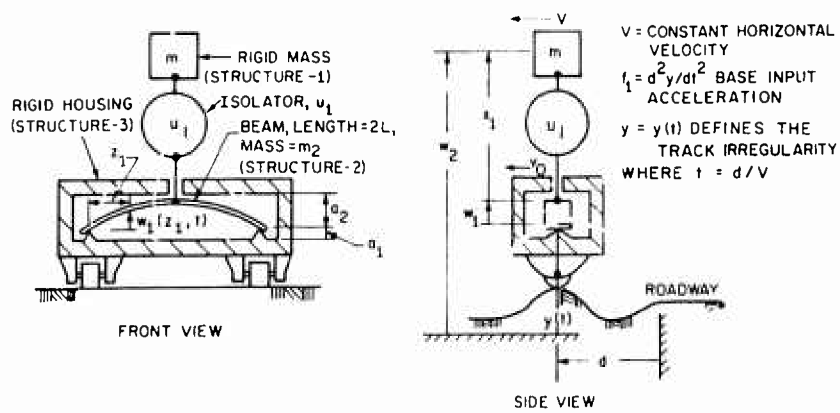
$$u_1(x_1, \dot{x}_1;\ k, c) = kx_1 + c\dot{x}_1.$$

The desired expressions result from the solution of the equations of motion which requires that the input $f_1(t)$ be specified. This problem is considered further in Example 2.

Example 2

GENERAL ISOLATION SYSTEM PROBLEM; TIME-OPTIMAL SYNTHESIS

We will consider a vehicle traversing a bump in the roadway as shown.



This is the same dynamic system as in Example 1, except that we are concerned with the details of the isolator but consider u_1 to be an unknown function of time $u_1(t)$. The performance index and response constraint are the same as in Example 1, except that each is viewed as a function of u_1 as well as t.

Performance Index

$$\psi = \max_t |\ddot{w}_2|.$$

Response Constraint

$$C_1(t, u_1) = w_1(L, t)$$

with

$$C_1^L = a_1; \quad C_1^U = a_2$$

and

$$C_2(t, u_1) = x_1(t) = w_2(t) - w_1(L, t) - y(t)$$

with

$$C_2^L = a_3; \quad C_2^U = a_4.$$

Parameter constraints do not enter into the time-optimal synthesis problem. Because of the relatively more simple form of the equations of motion for this formulation, we will carry out some of the details for arbitrary $f_1(t)$ avoided in Example 1.

In establishing the equations of motion, the isolator force $u_1(t)$ is treated as an external force for the system element on which it acts. Using conventional thin-beam theory, we find the equations of motion to be

$$\mathcal{L}_1 \equiv EI \frac{\partial^4 w_1}{\partial z_1^4} + \rho \frac{\partial^2 w_1}{\partial t^2} = u_1(t)\delta(z_1 - L) - m_2\ddot{y}(t)$$

$$\mathcal{L}_2 \equiv m \frac{d^2 w_2}{dt^2} = -u_1(t),$$

subject to the initial conditions

$$w_1(z_1, 0) = \frac{\partial w_1(z_1, 0)}{\partial t} = 0$$

$$w_2(0) = f(0); \quad \frac{dw_2}{dt} = \dot{f}(0)$$

and the boundary conditions

$$w_1(0, t) = w_1(2L, t) = \frac{\partial^2 w_1(0, t)}{\partial z_1^2} = \frac{\partial^2 w_1(2L, t)}{\partial z_1^2} = 0.$$

In these equations, EI is the stiffness of the beam, ρ its mass per unit length, $m_2 = 2L\rho$ its total mass, and m is the package mass. The isolator force $u_1(t)$, concentrated at midspan, is represented in terms of the Dirac delta function $\delta(z_1 - L)$.

These equations are in the form of Eq. (4.1). The system is quasi-linear, since the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are linear in the state variables w_1 and w_2 and the functions G_1 and G_2 are linear in u_1 and f_1. The response and constraint functions are linear in these variables also and will be evaluated in terms of u_1 and f_1 using the superposition forms of Eq. (4.10).

We begin by determining the response of the beam and the package mass to a unit force $\delta(t)$ applied at the isolator attachment points. Using the notation of Eq. (4.10), we can verify that R_{rj} and R_{kj} are given by

$$R_{11}(t-\tau) = \ddot{w}'_2(t-\tau)$$

$$R_{kj}(t-\tau) = \begin{cases} w'_1(L, t-\tau); & k = 1 \\ w'_2(t-\tau) - w'_1(L, t-\tau); & k = 2, \end{cases}$$

where

$$w'_1(L, t) = \frac{2}{m_2\Omega} \sum_{n=1,3,5}^{\infty} n^{-2} \sin(n^2\Omega t)$$

$$\Omega = \left(\frac{\pi}{2L}\right)^2 \left(\frac{EI}{\rho}\right)^{1/2}$$

$$w'_2(t) = \frac{-t}{m}$$

$$\ddot{w}'_2(t) = -\frac{\delta(t)}{m_2}.$$

The responses to the input $f_1(t)$, h_{s0} and C_{k0} in Eq. (4.10), are found to be

$$h_{10} = 0$$

$$C_{10}(t) = -\frac{8}{\pi\Omega} \int_0^t \sum_{n=1,3,\mathrm{odd}}^{\infty} (-1)^{(n-1)/2} n^{-3} f(\tau) \sin n^2\Omega(t-\tau)\, d\tau$$

$$C_{20}(t) = y(0) + \dot{y}(0) - C_{10}(t) - y(t)$$

Finally, the response and constraint functions are

$$h_1(t, u_1) = -\frac{u_1(t)}{m}$$

$$C_1(t, u_1) = C_{10}(t) + \int_0^t w'_1(L, t-\tau) u_1(\tau)\, d\tau$$

$$C_2(t, u_1) = C_{20}(t) + \int_0^t [w'_2(t-\tau) - w'_1(L, t-\tau)] u_1(\tau)\, d\tau.$$

Chapter 5

LIMITING PERFORMANCE CHARACTERISTICS OF SHOCK ISOLATION SYSTEMS

We consider in this chapter methods for determining the lower bound to the minimum performance index of a general shock isolation system without regard to any particular type of isolator element. The relationship between the bound and the constraint values is termed the *limiting performance characteristic* of the system (Chapter 4). We assume that the overall system is specified except for one or more isolator elements and that the input is known either in terms of specific waveforms (one or more) or by the description of a class of waveforms. Although explicit forms are provided only for rattlespace and peak acceleration criteria, the methods of solution are not limited to these choices.

5.1 Completely Described Environment

5.1.1 Single-Degree-of-Freedom Systems

Peak Acceleration and Rattlespace Criteria

Problem Formulation–The simplest SDF system to be studied is shown in Fig. 5.1. Regardless of the type of isolator element (e.g., spring or dashpot) under consideration, for our purposes we assume that the net force across the isolator is an unknown function of time $u(t)$.† The equation of motion for the rigid mass is

$$m\ddot{z} + u(t) = 0 \tag{5.1}$$

with the kinematic condition

$$z(t) = x(t) + f(t) \tag{5.2}$$

in the interval of interest, $t_0 \leqslant t \leqslant t_f$. Appropriate initial conditions on z and $\dot{z}$ at $t = t_0$ are specified. Unless otherwise indicated, we assume that the mass starts from rest at $t_0 = 0$ so that

†This is a nonrestrictive assumption insofar as the actual type of isolator employed. In particular, this formulation retains all interaction effects of the isolator on other portions of the system.

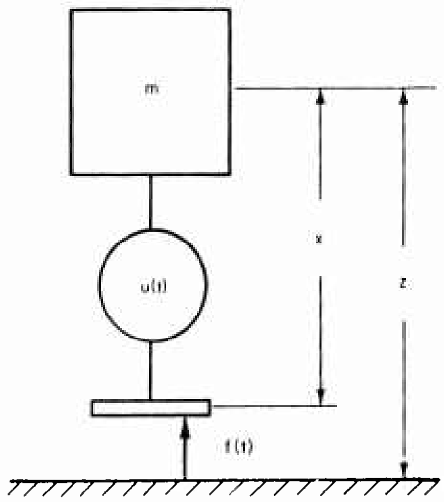


Fig. 5.1. Generic SDF system.

$$z(0) = \dot{z}(0) = 0. \tag{5.3}$$

Inasmuch as $u(t)$ represents the net force on the mass, arrangements more complicated than those shown in Fig. 5.1 are included. For example, in Fig. 5.2 the mass is acted upon by an external force $f_1(t)$ and supported by two isolators in parallel; the force across one being expressed in terms of the relative state variables $u_0(x, \dot{x})$ and in the other by a function of time $u_1(t)$. The quantity $u(t)$ in Eq. (5.1) in this case has the form

$$u(t) = u_1(t) + u_0(x, \dot{x}) - f_1(t),$$

as shown in Fig. 5.2.

Although the dynamic programming solution is unrestricted as to choice of performance index and constraint, the other methods considered require that these criteria be linear forms of the state variables. For the present, therefore, we shall treat linear forms and, in particular, choose the peak acceleration of the mass and the rattlespace as performance criteria. That is, either one may be the performance index, and the other the constraint. The absolute-value forms of these criteria are not, strictly speaking, linear but, for our purposes, we may accept them as such. The two optimization problems to be considered are, therefore

Problem 1

Performance Index: Peak Acceleration

$$\psi = \max_t |\ddot{z}|; \quad \min \psi = \psi^* = A \tag{5.4}$$

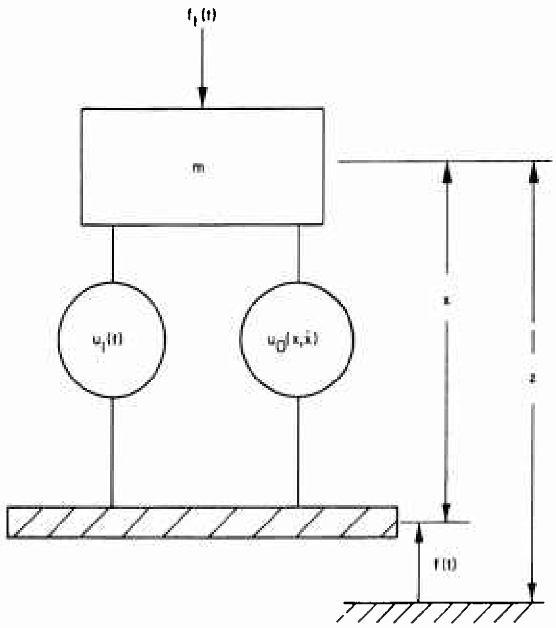


Fig. 5.2. An SDF system equivalent to the generic system.

Constraint: Rattlespace

$$\max_t |x| \leqslant D; \quad D \text{ prescribed.} \tag{5.5}$$

Problem 2

Performance Index: Rattlespace

$$\psi = \max_t |x|; \quad \min \psi = \psi^* = D \tag{5.6}$$

Constraint: Peak Acceleration

$$\max_t |\ddot{z}| \leqslant A; \; A \text{ prescribed.} \tag{5.7}$$

These two problems are reciprocal in the sense that if, in Problem 1, the minimum ψ for a rattlespace constraint D is found to be the peak acceleration

A, then an acceleration constraint of magnitude A in Problem 2 will lead to the minimum rattlespace D. (Note that this implies a single-valued relationship.) Therefore, we need use only one of these two problems to consider methods of solution. It is somewhat more convenient to select Problem 2, although on occasion we will deal with Problem 1.

A complete statement of Problem 2 is as follows: Given a prescribed base motion $f(t)$ and Eqs. (5.1), (5.2), and (5.3), which relate the state variables and the isolator force, find the force $u(t)$ such that the inequality (5.7) is satisfied and the performance index ψ defined in Eq. (5.6) takes on a minimum value. Let

$$\min \psi = \psi^* = \min_u \max_t |x| = D. \tag{5.8}$$

Then the point (A, D) lies on the limiting performance characteristic as shown in Fig. 5.3. The complete curve is found by repeated solutions for different values of A. For each point (A, D), the associated optimum isolator force is denoted by $u^*(t)$. This information is required in the indirect synthesis method.

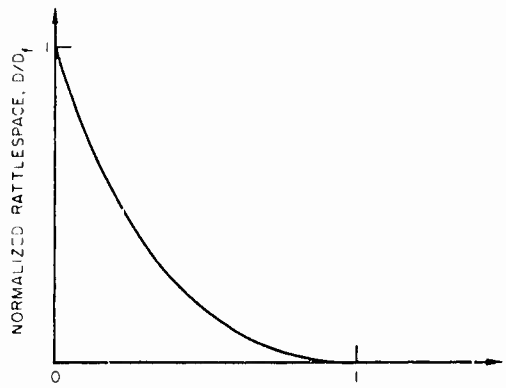


Fig. 5.3. Normalized form of limiting performance characteristic.

The limiting performance characteristic (Fig. 5.3) is shown in normalized form relative to the maximum displacement D_f and maximum acceleration A_f of the input disturbance. In this form, the intercepts are the points (1, 0) and (0, 1) which correspond to the extreme situation of a rigid connection and a constant-force isolator, respectively. In other words, as the rattlespace constraint approaches zero, the optimum isolator will transmit an acceleration approaching that of the base motion. Alternatively, as the rattlespace constraint approaches the base displacement, the optimum isolator will transmit a vanishingly small

acceleration. While we are interested practically in isolator performance between these limits, it may be noted that nonoptimum isolators easily can exceed them. For example, a sufficiently stiff, but not rigid, linear spring will transmit twice the peak base acceleration, whereas the mass could undergo an excursion far in excess of the base displacement if a resonant condition develops.

Analytical Solutions–The optimum isolation problem as we have posed it belongs to the class of variational problems that traditionally is approached analytically by the methods of the calculus of variations. The maximum-value form of the performance index and constraint is not suited to this classical method, however. Indeed, the calculus of variations can tell us nothing more than that the optimum isolator force $u^*(t)$ is of the bang-bang type, i.e., piecewise constant in time [4, 16-18]. It can lead neither to a quantitative analytical form for $u^*(t)$ nor to the minimum value of the performance index, and thus will not be considered further.

An analytical expression for the limiting performance characteristic can be found by direct means for an impulsive loading. The impulse loading also is a useful approximation to short-duration impacts, which are encountered frequently. Here A_f is not defined, but the impulse is given by

$$\lim_{t_f \to 0} \int_{t_0}^{t_f} \ddot{f}(t)dt = \dot{f}_0 = V. \tag{5.9}$$

This is equivalent to the base undergoing an initial velocity V. Thus, the displacement increases linearly with time and D_f is unbounded.

For the statement of Problem 2, energy considerations [1] lead to the relationship

$$AD = \frac{1}{2}V^2 \tag{5.10}$$

for the limiting performance characteristic. This is a rectangular hyperbola and is plotted in Fig. 5.4. Since neither D_f nor A_f is defined, the normalized plot cannot be used. The impulse case offers a convenient means of describing a useful graphical solution technique. We continue with the impulse loading case, deriving Eq. (5.10) and describing the character of the optimum isolator force $u^*(t)$.

Graphical Solution–Assume for the moment that, on application of the impulse, the isolator imparts the maximum allowable force to the mass. For unit mass ($m = 1$), this force is equal in magnitude to the constraint value of acceleration A and opposite in sign (Eq. (5.1)). The acceleration, velocity, and displacement for both the mass and base are shown in Fig. 5.5. From Eq. (5.2), $\dot{x}^* = \dot{z}^* - \dot{f}$, or

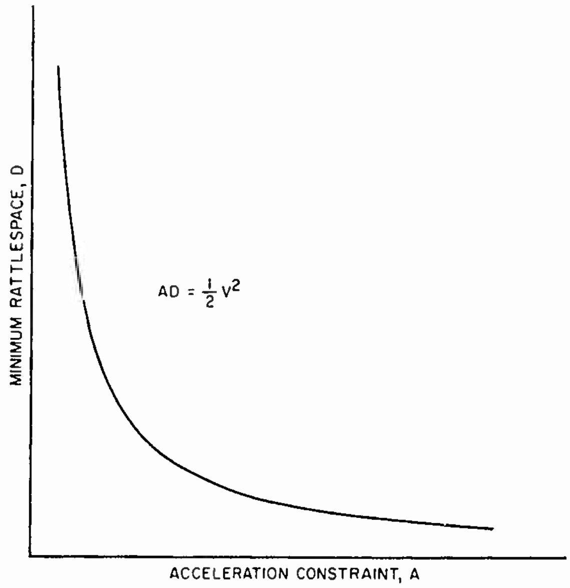


Fig. 5.4. Limiting performance characteristic for impulse loading, V.

$$x^* = \int_0^t (\dot{z}^* - \dot{f})dt,$$

so that the relative displacement of the mass is numerically equal to the area between the $\dot{z}^*$ and $\dot{f}$ curves in Fig. 5.5. The time at which $\dot{z}^* = \dot{f}$ is denoted by t_1.

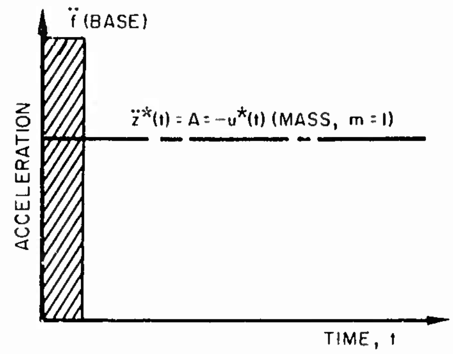
We are seeking the minimum rattlespace consistent with the peak acceleration constraint. From the geometry of Fig. 5.5, this is equivalent to finding the $\dot{z}^*$ curve whose slope does not exceed A in magnitude and for which the area between it and the $\dot{f}$ curve is a minimum. Up to t_1, it is clear that $|u(t)| = A$ accomplishes this, since (a) if, anywhere in the time interval $0 \leqslant t \leqslant t_1$, the slope of the $\dot{z}^*$ curve were less than A, a larger area would be enclosed, and (b) if a smaller area were enclosed, then the slope of the $\dot{z}^*$ curve would have to exceed A somewhere in the time interval. Moreover, beyond t_1 the relative displacement must decrease and, since we are free to select any $u(t)$ for $t > t_1$, the displacement at t_1 becomes the minimum rattlespace requirement D. Thus,

$$V = At_1 \quad \text{and} \quad D = \frac{1}{2} Vt_1$$

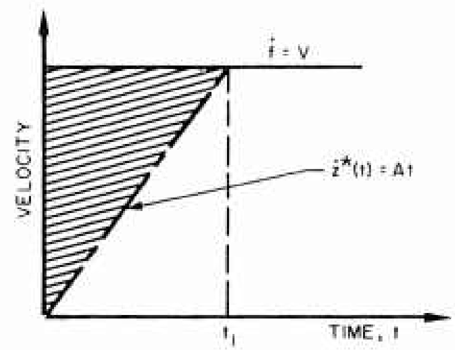
and

$$AD = \frac{1}{2} V^2,$$

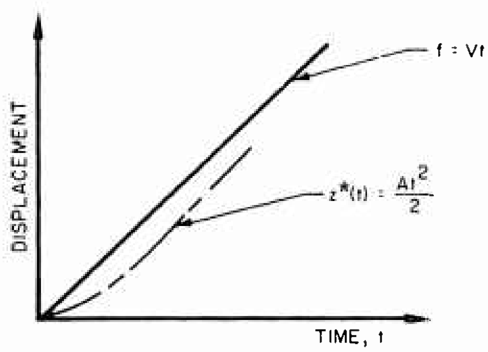
which is the result previously obtained in Eq. (5.10).



(a) Acceleration curves



(b) Velocity curves



(c) Displacement curves

Fig. 5.5. Base and optimum mass motions for impulse loading.

The graphical construction of Fig. 5.5 clearly suggests the nonunique character of the optimum isolator force beyond the time t_1 at which the base and mass reach equal speeds. While this does not influence the determination of the limiting performance characteristic, it is of consequence in the indirect synthesis method. Here, additional requirements can be imposed to fully define the isolator motion. It may be required, for example, that the mass return to its initial position relative to the base at some later time. The optimum motion in this case is more restrictive, but still not unique. A particular solution of the bang-bang type is shown in Fig. 5.6. The same result can be achieved, of course, with a continuous acceleration curve.

The graphical solution can be applied to other inputs. Figure 5.7 illustrates the procedure for an arbitrary base velocity $\dot{f}(t)$. Again the peak acceleration cannot exceed A, and the rattlespace D is to be minimized. The stepwise procedure is as follows:

Step 1. Construct a line from the origin with slope equal to the peak acceleration constraint A. This is the $\dot{z}^*$ trajectory, and its intersection with the $\dot{f}$ curve at Point 1 determines t_1. The area between the $\dot{f}$ curve and the line $\overline{01}$ is equal to the relative displacement x^* at t_1. Since $\dot{x}^*(t_1) = \dot{z}^*(t_1) - \dot{f}(t_1) = 0$, x^* has a relative maximum at t_1.

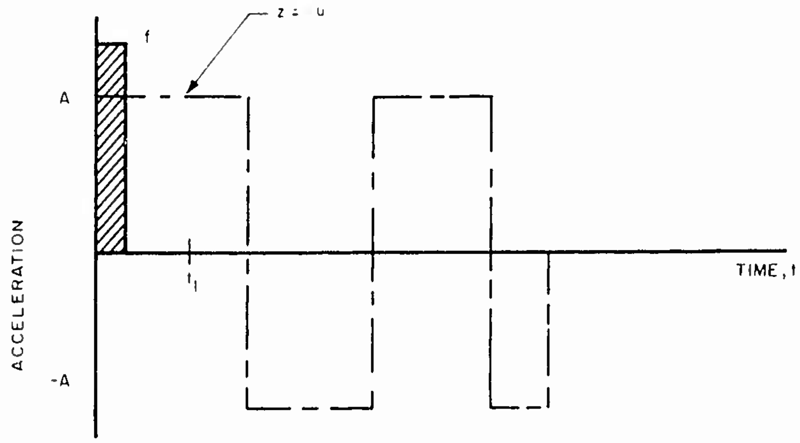
Step 2. Construct a line from Point 1 with slope $-A$. This is the continuation of the $\dot{z}^*$ trajectory: its intersection with the $\dot{f}$ curve at Point 2 determines time t_2. Compute the area between the $\dot{f}$ curve and the line $\overline{12}$. Call this area Δx_2. Then, $x^*(t_2) = x^*(t_1) - \Delta x_2$. Since $\dot{x}^*(t_2) = 0$, $x^*(t_2)$ is a relative minimum.

Step 3. If $|x^*(t_1)| > |x^*(t_2)|$ and if none of the subsequent relative maxima or minima of x^* exceeds $|x^*(t_1)|$, then the minimum rattlespace is $D = |x^*(t_1)|$. If, however, $|x^*(t_1)| < |x^*(t_2)|$, go to Step 4. If a subsequent maximum or minimum of x^* exceeds $|x^*(t_1)|$, then go to Step 5.

Step 4. A modified $\dot{z}^*$ curve is constructed as the line $\overline{345}$. It has slope $-A$, and the area between the $\dot{f}$ curve and the line $\overline{034}$, $x^*(t_4)$, is equal to twice the area between the $\dot{f}$ curve and line $\overline{45}$. This requires a trial-and-error procedure. The minimum rattlespace is $D = |x^*(t_4)| = |x^*(t_5)|$, provided no subsequent maximum or minimum of x^* exceeds this value. If it does, go to Step 5.

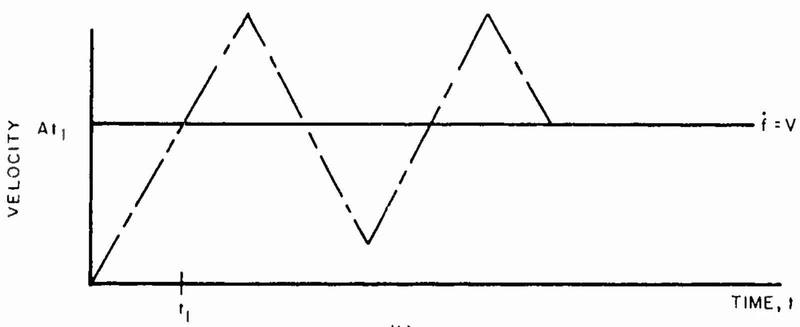
Step 5. This step need be followed only in those instances where the input disturbance is such that the rattlespace is determined at relatively late times. The general procedure of Step 4 is followed, to modify the appropriate segment of the $\dot{z}^*$ curve where the minimum rattlespace determined up to that time is exceeded. The reader who has carried out the construction is advised to construct an input velocity curve having this feature.

The $\dot{z}$ curve beyond the time at which the minimum rattlespace is established is, as mentioned earlier, not unique, and additional conditions need to be imposed. Since this is entirely a kinematic process, the graphical construction may



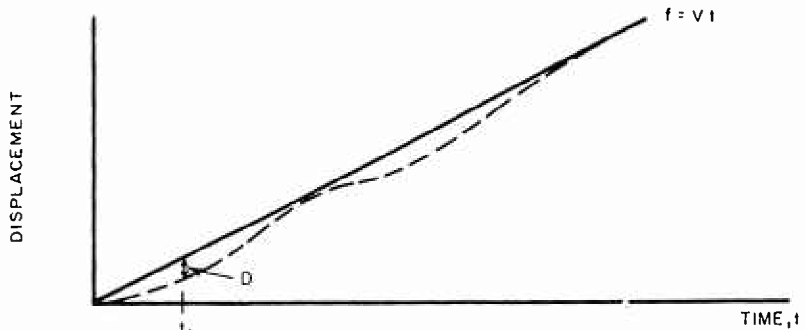
(a)

(a) Acceleration curves



(b)

(b) Velocity curves



(c)

(c) Displacement curves

Fig. 5.6. Optimum "bang-bang" restorative isolator for impulse loading.

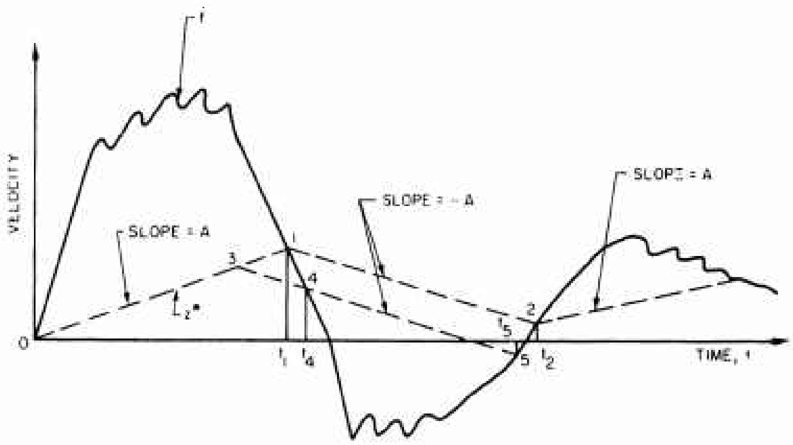


Fig. 5.7. Graphical solution for arbitrary input.

be applied. The optimum isolator force $u^*(t)$ is constructed from the derivative of $\dot{z}^*$ according to Eq. (5.1). The graphical procedure was used in Ref. 19 to compute limiting performance characteristics for several inputs occurring in ship design.

A graphical construction can be carried out for the Problem 1 formulation where the rattlespace is constrained and the peak acceleration is minimized [15,17]. The procedure is slightly less convenient, and in view of the reciprocal nature of Problems 1 and 2 we will not consider it further.

Numerical Solutions—Although the graphical approach is attractively simple and provides a certain insight into the nature of the optimum isolator response, it is difficult to systemize to encompass general disturbances and other optimization criteria. We seek, therefore, a more general solution procedure. In discrete form, the optimum isolator problem is recognized as belonging to a broader class of mathematical programming problems for which powerful computational techniques are available. When both the constraints and performance index involve linear functions of the state variables, a linear programming (LP) solution is appropriate. Otherwise, the more general method of dynamic programming must be used.

We will describe the discrete formulation and identify the conditions necessary for it to be one of linear programming. The reduction to standard LP form applicable to available LP computer codes is described in Appendix B. Where hand calculations are to be performed, the reader is advised to consult one of the many available references on linear programming [e.g., 21].

Let the time interval of interest be divided into $I - 1$ equal subintervals Δt. Denote the value of the variables t, x, z, and f at the beginning of the ith interval by t_i, x_i, z_i and f_i. Then

$$
\begin{aligned}
t_i &= (i-1)\Delta t \\
x_i &= x(t_i) \\
z_i &= z(t_i) \\
f_i &= f(t_i).
\end{aligned}
\tag{5.11}
$$

Assume that the isolator force is constant† during each subinterval of time (Fig. 5.8); then, in discrete terms, the equation of motion is

$$m\ddot{z}_i + u_i = 0, \tag{5.12}$$

subject to the kinematic condition

$$x_i = z_i - f_i \tag{5.13}$$

and the initial conditions

$$z_1 = \dot{z}_1 = 0. \tag{5.14}$$

The set of numbers $u_1, u_2, \ldots, u_{I-1}$ representing $u(t)$ is unknown. Finding the particular set u_i^* for which (Problem 2)

$$\max_i |\ddot{z}_i| \leqslant A \tag{5.15}$$

and

$$\psi = \max_i |x_i| \tag{5.16}$$

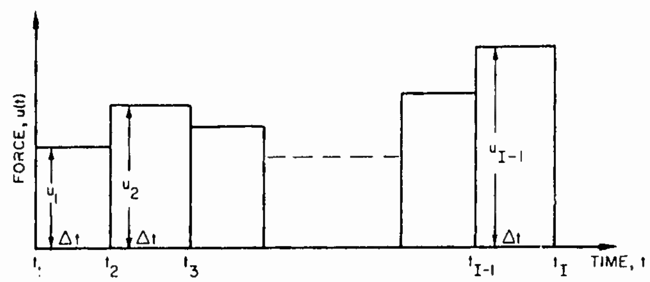


Fig. 5.8. Piecewise constant approximation of the isolator force.

†A piecewise linear approximation is considered in Appendix B.

is a minimum is a problem in linear programming, although not in standard form. It may be reduced to standard LP form by rewriting Eq. (5.15) as

$$-A \leqslant \ddot{z}_i \leqslant A; \quad i = 1, 2, \ldots, l \tag{5.17}$$

and observing that Eq. (5.16) is equivalent to requiring the x_i to satisfy the inequality constraints

$$-\psi \leqslant x_i \leqslant \psi; \quad i = 1, 2, \ldots, l. \tag{5.18}$$

The LP problem is now to find the u_i^* that minimizes ψ subject to the constraints of Eqs. (5.17) and (5.18). The x_i and $\ddot{z}_i$ are related to the u_i by Eqs. (5.12) and (5.13).

Dynamic programming is the name given by its inventor, Richard Bellman [22], to a computationally motivated procedure for solving optimization problems through a sequence of smaller problems. It is a very general method but becomes computationally impractical as the size of the problem increases. At present, the practical size limit for general nonlinear isolation systems appears to be two or three degrees of freedom. More complex systems will require improved computational procedures and greater computer hardware capabilities in both storage and speed of computation. However, dynamic programming is eminently practical for SDF systems and can accept any sort of nonlinear form for the constraints and the performance index.

The method will be described first with respect to the Problem 2 formulation (i.e., peak acceleration as the constraint and rattlespace as the performance index). It will then be generalized for arbitrary criteria functions. Although dynamic programming is essentially simple in concept (at least as applied here), the notation is a bit unwieldy and tends to complicate a first presentation. The calculation described in Example 3 should prove helpful in connection with the following description.

We are seeking the particular set of u_i, say u_i^*, for which $\max_i|\ddot{z}_i| \leqslant A$ and $\psi = \max_i|x_i|$ is a minimum. Let the *state* of the system at time t_i be described by the values of x_i and $\dot{x}_i$, or more briefly, by the state vector $\mathbf{x}_i$, where

$$\mathbf{x}_i = \begin{Bmatrix} x_i \\ \dot{x}_i \end{Bmatrix}. \tag{5.19}$$

Assume that the system is in a particular state at the start of the ith time interval, denoted by $\mathbf{x}_i^p$. At this time an admissible value of u_i, which causes the system to be in state $\mathbf{x}_{i+1}^p$ at the end of the interval, is selected. The relation between $\mathbf{x}_i^p$ and $\mathbf{x}_{i+1}^p$ is, in fact, the solution of Eq. (5.12). Now further assume

that the minimum value of ψ is known at t_{i+1} for all subsequent times and any admissible state p ($p = 1, 2, \ldots, P$). These values of min ψ will be denoted by $\phi(\mathbf{x}_{i+1}^p)$ and are given by

$$\phi(\mathbf{x}_{i+1}^p) \equiv \min_{u_q} \psi = \min_{u_q} \max |x_r| \tag{5.20}$$

for

$$q = i+1, \ldots, I-1$$

$$r = i+1, \ldots, I$$

$$p = 1, 2, \ldots, P$$

and where the optimization is over all admissible u_i as determined by the constraint Eq. (5.15).

Observe that $\phi(\mathbf{x}_{i+1}^p)$ is a two-dimensional array of $P = N_1 N_2$ entries, where N_1 and N_2 are, respectively, the number of discrete values admitted for the state variables x and $\dot{x}$. The pth entry in the ϕ matrix is the value of the minimum rattlespace which would result if the system started at time t_{i+1} from the initial state $\mathbf{x}_{i+1}^p$. The size of the ϕ matrix is necessary because any of the P admissible states at t_i may be the optimum state, the precise one remaining unknown until the end of the process. The desired value of min ψ is given by $\phi(x_1^1)$, where $p = 1$ corresponds to the prescribed initial state at t_1.

For each state p and admissible value of u_i, two estimates of rattlespace must be compared. One is the value of $|x_i^p|$, the selected relative displacement in state p, and the other is the appropriate entry in the $\phi(\mathbf{x}_{i+1}^p)$ matrix.† The minimum with respect to the u_i of the larger of these is the desired entry in the $\phi(\mathbf{x}_i^p)$ matrix; i.e.,

$$\phi(\mathbf{x}_i^p) = \min_{u_i} \max \left[|x_i^p|, \phi(\mathbf{x}_{i+1}^p) \right]. \tag{5.21}$$

This constitutes a recursive relationship which proceeds backward in time. The process is started at $i = I$ by noting that $\phi(\mathbf{x}_I^p) = x_I$ for all p, since the state of the system is known. The computation must be performed at each t_{i+1} for a sufficient range of P so that subsequently at t_i the resulting state $\mathbf{x}_{i+1}^p$ will be close enough to some $\mathbf{x}_{i+1}^p$ in the matrix of known $\phi(\mathbf{x}_{i+1}^p)$ values to permit accurate interpolation.

†Generally the calculated $\mathbf{x}_{i+1}^p$ will not correspond identically with a row and column of the ϕ matrix, making interpolation necessary (Example 3).

We may clarify matters notationally by relating the stage of the (computational) process to the state of the system. Up to this point, the state has been indicated by the superscript p. This notation may be dropped, since it is understood that all admissible states are considered at each stage of the process, and the stage is indicated by a subscript on ϕ consistent with that on the state variables. Then Eq. (5.21) is written as

$$\phi_{I-i+1}(\mathrm{x}_i) = \min_{u_i} \max\left[|x_i|, \phi_{I-i}(\mathrm{x}_{i+1})\right]; \quad i = I-1, I-2, \ldots, 1 \qquad (5.22)$$

with

$$\phi_1(\mathrm{x}_I) = x_I.$$

The constraint condition is satisfied by restricting the minimization in Eq. (5.21) to admissible values of the u_i. At the completion of the process

$$\phi_I(\mathrm{x}_1) = \phi_I(0) = \psi^*,$$

where $\mathrm{x}_1 = 0$ corresponds to the specified initial state. At the same time the minimum among the values of u_1 considered is determined. This is the piecewise constant approximation to $u^*(t)$ during the first interval $0 \leqslant t \leqslant \Delta t$, and is designated as u_1^*. Using this value of u_1^*, we compute x_2^* and $\dot{x}_2^*$ using Eqs. (5.12), (5.13), and (5.14) with the index i now going forward in time. Then u_2^* is the minimum u_2 associated with the prior calculation for $\phi_{I-1}(\mathrm{x}_2^*)$. This is obtained by interpolation from the ϕ_{I-1} matrix. It is not necessary to retain both u_2^* and $\phi_{I-1}(\mathrm{x}_2^*)$ values, since min u_2 is quickly redetermined from Eq. (5.22) if the $\phi_{I-1}(\mathrm{x}_2^*)$ matrix has been saved. In this manner, the optimum isolator force and state variable trajectories are determined stage by stage, proceeding forward in time.

Two important observations are to be made from Eq. (5.22). From a computational point of view, note that we have replaced the original mathematical programming formulation, which required the solution of an I-dimensional minimization problem, by a sequence of I one-dimensional minimizations. This is the essence of the dynamic programming approach. From a more fundamental point of view, Eq. (5.22) implies that, for the solution to be optimum, each stage of the process must proceed optimally relative to the state of the system at the prior stage. This is a loose translation of Bellman's *Principle of Optimality* [22] which, in more of the language of dynamic programming, asserts that regardless of the decisions made in reaching a particular state, subsequent decisions must constitute an optimal solution with respect to this state. Here, "decision" refers to the choice of u_i. This principle allows us to generalize Eq. (5.22).

The power of dynamic programming lies in the simplicity of the computation, its lack of dependence on the functional form of the constraints or the performance index, and the fact that constraints work to one's advantage as they reduce the state space that must be searched. Offsetting these desirable features, however, is the overpowering effect of the number of computations increasing factorially with the addition of each state variable. The literature does not contain results for other than SDF systems, for which dynamic programming is quite reasonable. It would appear that two- or perhaps three-degree-of-freedom systems (six state variables) could be handled with available techniques and computers. Beyond that, more sophisticated computational procedures and computer capabilities are required.

Arbitrary Optimization Criteria

When the performance index, constraints, or both are nonlinear in the state variables, only the method of dynamic programming is applicable. Let the constraints and performance index be arbitrary functions of the state variables. In particular, the performance index ψ is taken to be the maximum over the time of some response function $h(\mathbf{x}, u)$. In discrete form,

$$\psi = \max_i |h(\mathbf{x}_i, u_i)|.$$

Then, by virtue of the Principle of Optimality, the relationship between the optimum performance index from one stage of the process to the next is

$$\phi_{I-i+1}(\mathbf{x}_i) = \min_{u_i} \max\left[|h(\mathbf{x}_i, u_i)|, \phi_{I-i}(\mathbf{x}_{i+1})\right]. \tag{5.23}$$

This is an obvious generalization of Eq. (5.22).

The process starts with

$$\phi_1(\mathbf{x}_i) = \max_t |h(\mathbf{x}, u)|; \qquad t > t_I \tag{5.24}$$

and is carried out recursively for $i = I - 1, I - 2, \ldots, 1$. The resulting $\phi_I(\mathbf{x}_1)$ is the desired minimum performance index, where $\mathbf{x}_1$ is the specified initial state. The choice of admissible states and the range of u_i considered in the evaluation of Eq. (5.23) must be consistent with the imposed constraints. The equations of motion governing the system dynamics are used to determine a new state at the next time step for a particular choice of u_i. A second pass with Eq. (5.23), forward in time, is required to establish the optimum isolator force and state-variable trajectories, as described previously. Repetitive solutions for different constraint levels yield the limiting performance characteristics.

Representative Results

Limiting performance characteristics for peak acceleration and rattlespace criteria are determined in Refs. 3 and 15 for a variety of input waveforms. Representative results are shown in Figs. 5.9 through 5.12. While all inputs are not defined quantitatively, they are shown to scale in the inserts of the figures. Shown for comparison purposes are the optimum performance characteristics for a linear spring-dashpot isolator [15]. These results were obtained by methods discussed in Chapter 6.

There has been some discussion in the literature of an "early-warning" or "preview" type of isolator; i.e., an active device which senses certain details of the impending input disturbance. Such an isolator may be thought of as being additionally optimized with respect to the initial conditions x_1 and $\dot{x}_1$, whose range depends on the preview "length" or "warning time." This is equivalent to assigning an appropriate lead time to the input waveform and starting the system from rest at the earlier time. The limiting performance characteristics for a preview isolator are compared with those of a nonpreview isolator in Fig. 5.13. We see that a substantial improvement in performance is possible for an isolator with preview control.

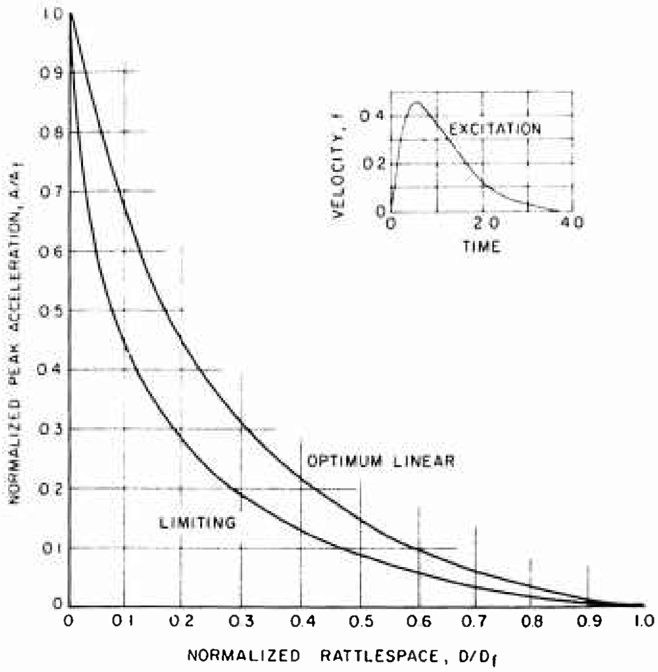


Fig. 5.9. Optimum linear and limiting performance characteristic for an SDF system.

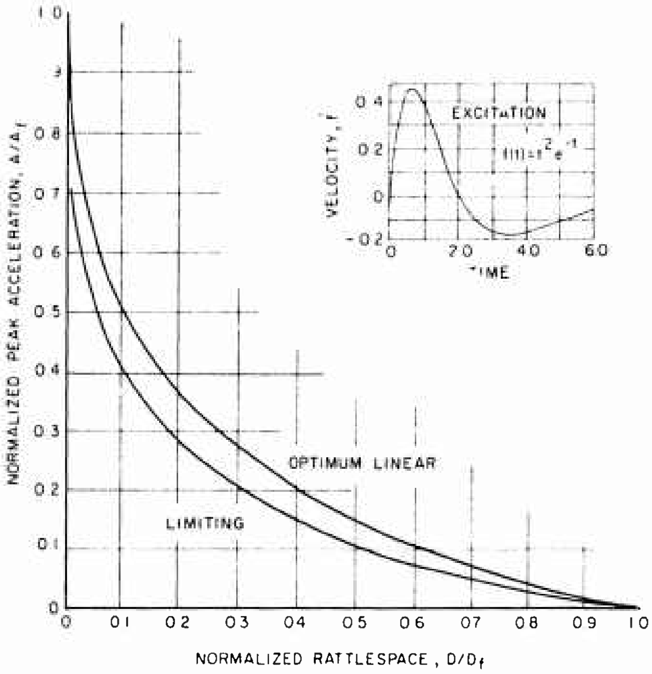


Fig. 5.10. Optimum linear and limiting performance characteristic for an SDF system.

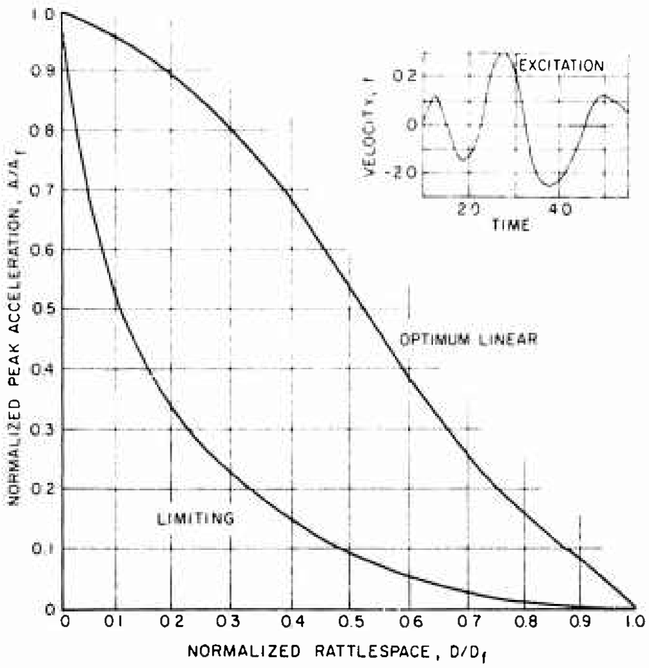


Fig. 5.11. Optimum linear and limiting performance characteristic for an SDF system.

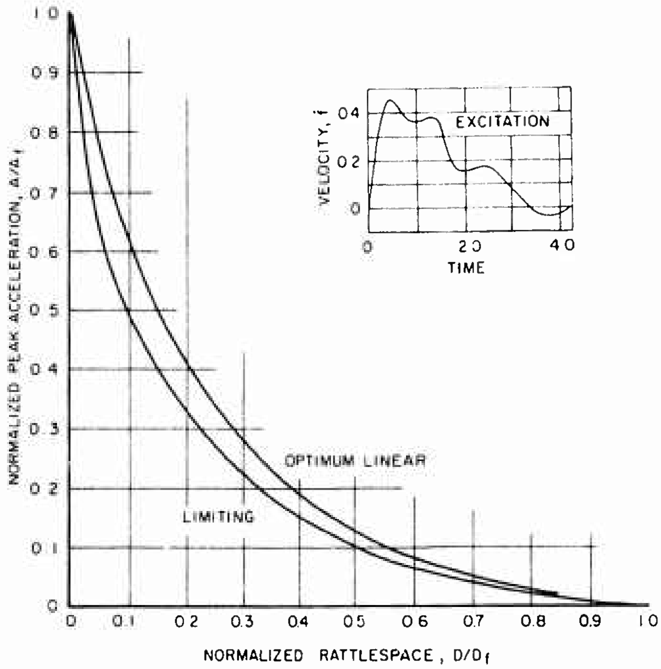


Fig. 5.12. Optimum linear and limiting performance characteristic for an SDF system.

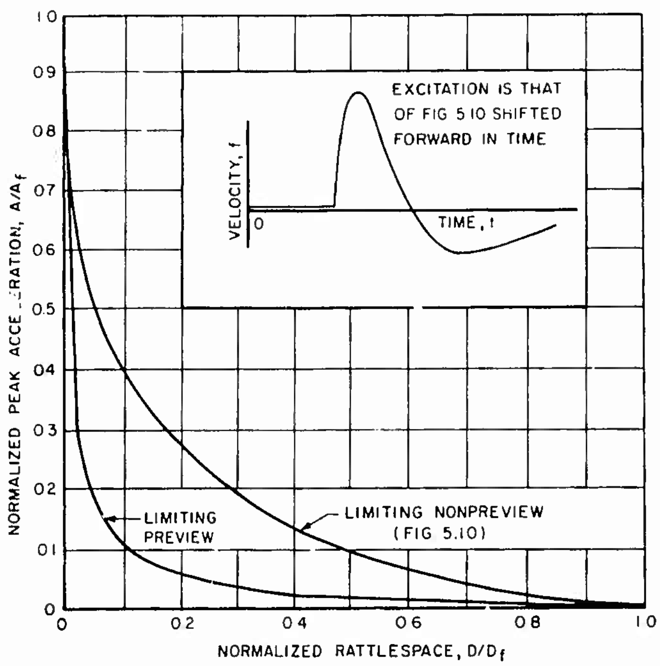


Fig. 5.13. Limiting performance characteristic of preview isolator.

5.1.2 Multiple Degree-of-Freedom Systems

The extension from SDF to MDF systems does not complicate the general formulation for the limiting performance characteristic. However, the computational effort required for solution becomes impractical with increasing system complexity (number of degrees of freedom). The method of dynamic programming remains generally applicable but probably would not be attempted for systems larger than about two or three degrees of freedom because of present limitations in high-speed computer storage. We will consider a special class of MDF systems, termed quasi-linear systems, for which linear programming solutions are possible. While relatively complex systems can be handled by presently available LP codes, the effort can be substantial and special-purpose programs are required to manage the data. Thus, while it is a simple matter to formulate the method of approach, it may be that not every reader will be in a position to carry out the solutions.

Quasi-Linear Systems

A *quasi-linear system* is defined as one that responds as a linear system when the isolator forces are replaced by explicit, albeit unknown, functions of time. In addition, it is required that the performance index and all constraints be linear functions of the state variables. We will outline the LP formulation for determining limiting performance characteristics for a general quasi-linear system with a maximum over time performance index. When the system has more than several degrees of freedom, it is expected that the automated methods would be used rather than proceeding directly from the equations of motion.

We will consider an isolation system possessing a finite number of degrees of freedom and which contains J unspecified isolator elements. The force (or moment) in the jth isolator is again denoted by u_j and considered to be an explicit function of time $u_j(t)$. In addition, L external input waveforms are specified, a typical one being denoted by $f_\ell(t)$. If the system has P degrees of freedom, there will be $2P$ state variables (displacements and velocities) to consider, of which, $2J$ $(J \leqslant P)$ will refer to the state of the unknown isolators relative to their attachment points.

Since the prescribed elements of the system are assumed to be linear, the equations of motion can be expressed as a $2P$ set of first-order, ordinary,† linear, differential equations. In matrix notation this may be written

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{F}, \tag{5.25}$$

where $\mathbf{x}$ is the state vector with $2P$ components, $\mathbf{u}$ is a $2P$-element isolator force vector with at most J nonzero elements, $\mathbf{F}$ is a $2P$-element input vector with at

†The formulation can be extended to continuous system elements represented by partial differential equations in an obvious manner.

most L nonzero elements, and **A** and **B** are both $2P$ square matrices appropriate to the system under consideration. A number of constraints K are imposed on the system response. These may be functions of the state variables (e.g., stresses, displacements, accelerations) and are denoted by C_k. By virtue of Eq. (5.25), these may be considered functions of time and the unknown isolator forces u_j. Thus, general constraint functions are written as

$$C_k^L \leqslant C_k(t, u_j) \leqslant C_k^U; \quad k = 1, 2, \ldots, K, \tag{5.26}$$

where the $2K$ constants C_k^L and C_k^U are prescribed.

As the performance index we choose the maximum of any one of S different (but comparable) response functions (Chapter 2). While these functions, h_s, usually involve the state variables, we consider them to be explicit functions of time and the isolator forces. Then the performance index is written as

$$\psi = \max_s \max_t |h_s(t, u_j)|; \quad s = 1, 2, \ldots, S.$$

Since we seek to minimize ψ, this expression is equivalent to imposing the additional constraints

$$|h_s(t, u_j)| \leqslant \psi \text{ for all } t \text{ and } s = 1, 2, \ldots, S. \tag{5.27}$$

We now replace the continuous functions by a discrete representation (Eq. (5.11)). The only change in notation is that the normally subscripted variables receive a second subscript to denote time; e.g., $u_j(t_i)$ is written u_{ji}.

The performance index becomes

$$\psi = \max_s \max_i |h_s(t_i, u_{ji})|; \quad \begin{array}{l} s = 1, 2, \ldots, S \\ i = 1, 2, \ldots, I. \end{array} \tag{5.28}$$

The problem is to find the u_{ji} ($j = 1, 2, \ldots, J$; $i = 1, 2, \ldots, I$) such that ψ is minimized and the constraints

$$|h_s(t_i, u_{ji})| \leqslant \psi; \quad \begin{array}{l} j = 1, 2, \ldots, J \\ s = 1, 2, \ldots, S \\ i = 1, 2, \ldots, I \end{array} \tag{5.29}$$

$$C_k^L \leqslant C_k(t_i, u_{ji}) \leqslant C_k^U; \quad k = 1, 2, \ldots, K$$

are satisfied.

Since the u_{ji} are related linearly to the $\mathbf{x}_i$ through the solution of Eq. (5.25) and both the performance index and constraints also involve the u_{ji} linearly, the problem is recognized as one of linear programming. If the solution is repeated

for different levels of the constraint functions, the resulting relationship between min ψ and the constraints provides the desired limiting performance characteristic.

From a computational point of view, the size of the LP problem depends on the number of unknown u_{ji} and the number of constraint functions. The number of degrees of freedom of the system governs the rank of Eqs. (5.25) and is a factor only in finding this solution. The coefficient of the u_{ji} in the constraint expressions, of course, depends on this solution, but is constant insofar as the LP problem is concerned. Thus, Eq. (5.25) has to be solved only one time, which means that the size of the LP problem is independent of the number of degrees of freedom of the system. Instead, it depends on three factors; (a) the number of unknown isolator forces J, (b) the number of constraint functions $K + S$, and (c) the number of subdivisions of the time interval of interest I. Thus, the computational effort involved in finding the limiting performance characteristic is not influenced by the kinematics of the problem.

The detailed conversion to standard LP form, acceptable to available LP codes, is discussed in Appendix B. Reference 3 describes a linear programming preprocessor code which provides various options for h_s, C_k, and u_j; h_s and C_k being rather arbitrary functions which could represent stresses, displacements, velocities, forces, or practically any other physical quantity of interest to the system that can be expressed as a linear function of the state variables. Constraints can be imposed on any of the state variables or on the isolator forces, singly or in combination. Those on the isolator forces can be in terms of magnitude or rise time.

To illustrate these general procedures in a specific case, we consider the two-degree-of-freedom, two-isolator system shown in Fig. 5.14 and subjected to a base displacement $f(t)$. Constraints are placed on the two relative displacements x_1 and x_2, and we require that the maximum acceleration transmitted to either mass be minimized. In terms of the general notation, the constraint functions are

$$C_1 = x_1; \quad C_2 = x_2,$$

and we require that

$$\begin{aligned} D_1^L &\leqslant x_1 \leqslant D_1^U \\ D_2^L &\leqslant x_2 \leqslant D_2^U. \end{aligned} \tag{5.30}$$

The upper and lower displacement bounds D_1^L, D_1^U, D_2^L, D_2^U are prescribed constants.

The response functions that make up the performance index are

$$h_1 = \ddot{z}_1; \quad h_2 = \ddot{z}_2.$$

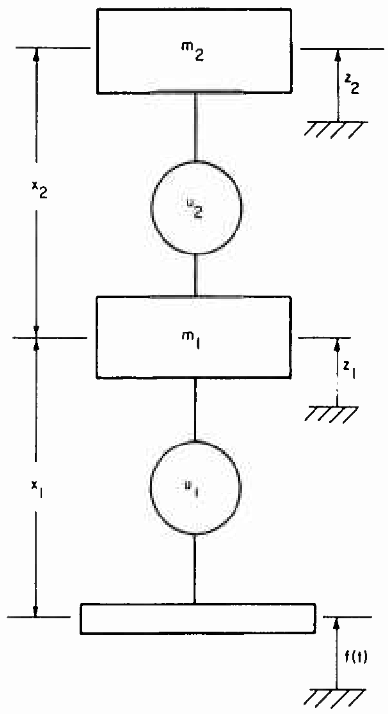


Fig. 5.14. Two-degree-of-freedom, two-isolator system.

Therefore, the performance index is

$$\psi = \max_s \max_t |\ddot{z}_s|; \qquad s = 1, 2$$

or

$$\psi = \max[\max_t |\ddot{z}_1|, \max_t |\ddot{z}_2|] . \tag{5.31}$$

It is now necessary to express the state variables in terms of the isolator forces u_j. The equations of motion are

$$\begin{aligned} m_1 \ddot{z}_1 + u_1(t) - u_2(t) &= 0 \\ m_2 \ddot{z}_2 + u_2(t) &= 0, \end{aligned} \tag{5.32}$$

with the kinematic conditions

$$\begin{aligned} x_1 &= z_1 - f \\ x_2 &= z_2 - z_1 \end{aligned} \tag{5.33}$$

and the initial conditions

$$z_1(0) = \dot{z}_1(0) = z_2(0) = \dot{z}_2(0) = 0.$$

Equations (5.32) can be written as a set of four first-order differential equations in the matrix form (Eq. (5.25))

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{F},$$

where

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} \qquad \mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{Bmatrix} \qquad \mathbf{F} = \begin{Bmatrix} 0 \\ 0 \\ -\ddot{f} \\ 0 \end{Bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\dfrac{1}{m_1} & \dfrac{1}{m_1} & 0 & 0 \\ \dfrac{1}{m_1} & \left(-\dfrac{1}{m_1} + \dfrac{1}{m_2}\right) & 0 & 0 \end{bmatrix},$$

and x_3 and x_4 are the velocities $\dot{x}_1$ and $\dot{x}_2$, respectively. The solution to these for x_1 and x_2 is

$$\begin{aligned} x_1(t) &= -f(t) - \frac{1}{m_1}\int_0^t (t-\tau)[u_1(\tau) - u_2(\tau)]d\tau \\ x_2(t) &= \frac{1}{m_1}\int_0^t (t-\tau)[u_1(\tau) - u_2(\tau)]d\tau - \frac{1}{m_2}\int_0^t (t-\tau)u_2(\tau)d\tau. \end{aligned} \tag{5.34}$$

Discretization of these expressions depends on the form adopted for $u_j(t)$. For a piecewise constant approximation to u_j, each of the integral terms

$$x(t) = \frac{1}{m}\int_0^t (t-\tau)u_j(t)d\tau$$

becomes

$$x_i = \frac{(\Delta t)^2}{2m} \sum_{n=1}^{i-1} (2i - 2n - 1) u_{ji} . \tag{5.35}$$

As the complexity of the system increases, the assembly of the **A** and **B** matrices in Eq. (5.25) becomes increasingly cumbersome, and automated methods are required. The linear programming preprocessor considered in Ref. 3 develops Eq. (5.34) directly by superposing solutions corresponding to unit impulses placed sequentially at each of the isolator attachment points. The advantage of this approach lies in the fact that existing general-purpose structural dynamic codes can be used to generate unit-impulse responses numerically. Viewed in this fashion, Eq. (5.34) is recognized as the Duhamel (convolution) integral form of the solution to Eq. (5.32). More generally, for any quasi-linear system, the response and constraint functions can be written as

$$h_s(t, u_j) = h_{s0}(t) + \sum_{j=1}^{J} \int_0^t R_{sj}(t - \tau) u_j(\tau) d\tau$$

and (5.36)

$$C_k(t, u_j) = C_{k0}(t) + \sum_{j=1}^{J} \int_0^t R_{kj}(t - \tau) u_j(\tau) d\tau ,$$

where R_{sj} and R_{kj} are the appropriate system responses to a unit impulse at the attachment point of the jth isolator, and $h_{s0}(t)$ and $C_{k0}(t)$ the responses of that portion of the system, exclusive of the isolators, acted on by the input $f_\ell(t)$. To illustrate various features of the general formulation, we will consider several simple problems, Examples 4, 5, and 6.

5.2 Incompletely Described Environment

So far in this chapter it has been assumed that the input disturbance was a uniquely prescribed function of time. To be realistic, of course, the isolation system designer seldom knows the complete time details of the disturbance, and this was recognized in the general formulation of Chapter 4. We will restrict our consideration of uncertainty in shock inputs to a deterministic framework, though random disturbances are included in Chapter 8. We will consider the case of a finite number of waveforms (multiple inputs) and a class description containing an infinite number of input waveforms. Only quasi-linear systems will be considered where application to large systems is intended.

5.2.1 Multiple Inputs

A family of shock input waveforms is prescribed, denoted by $f_\ell(t)$; $\ell = 1, 2, \ldots, L$. This notation includes the possibility that (a) the system is equally likely to sustain any particular input separately, (b) two or more of the inputs act simultaneously at different (prescribed) points on the system, or (c) any combinations of input may or may not occur, with equal likelihood. Note that the assumption of simultaneity in (b) is not restrictive, since the time origin of each $f_\ell(t)$ always can be chosen to satisfy this condition while admitting any degree of time phasing.

For multiple inputs, the limiting performance characteristic refers to the least value of the performance index consistent with the constraints, whatever admissible combination of the L inputs occurs. This implies an upper bound to the minimum performance index associated with a "worst" input or set of inputs among those prescribed. Since the constraints must not be violated for any combination of inputs, the solution requires more than just the examination of each admissible input set. Otherwise the possibility that one set of inputs will cause the constraint to be operative while another leads to a minimum performance index might be overlooked.

Applicable solution techniques are dynamic programming and linear programming, the latter requiring the system to be quasi-linear and the optimization criteria to be linear. Since in most instances the dynamic programming solution is similar to that for the class description of inputs for which linear programming does not apply, only the LP solution is described here.

The equations of motion for a general quasi-linear system are given by Eq. (5.25). The performance index and constraints must be linear functions of the state variables and inputs, and are written to emphasize their dependence on $f_\ell(t)$. Thus, the problem of finding the limiting performance characteristic requires the selection of the J isolator forces $u_j(t)$ for which the constraints

$$C_k^L \leqslant C_k(t, u_j, f_\ell) \leqslant C_k^U; \qquad \begin{array}{l} \ell = 1, 2, \ldots, L \\ k = 1, 2, \ldots, K \end{array} \tag{5.37}$$

are satisfied and the performance index

$$\psi = \max_{\ell} \max_{s} \max_{t} |h_s(t, u_j, f_\ell)|; \qquad \begin{array}{l} \ell = 1, 2, \ldots, L \\ s = 1, 2, \ldots, S \end{array} \tag{5.38}$$

is minimized. As before, the problem is reduced to standard LP form by replacing Eq. (5.38) with the additional constraints

$$|h_s(t, u_j, f_\ell)| \leqslant \psi \tag{5.39}$$

for all t, ℓ, and s. In discrete form, where the time interval of interest is subdivided into I subintervals, Eq. (5.39) amounts to introducing $I \times L \times S$

constraint relations. The expression for the minimum performance index now reads

$$\psi^* \equiv \min_{u_j} \psi = \min_{u_j} \max_{\ell} \max_{s} \max_{t} |h_s|. \tag{5.40}$$

5.2.2 Input Class

We consider the input disturbance to be a member of a class which may include an infinite number of waveforms. The manner in which an input class is defined was discussed in Chapter 3 (Fig. 3.1); a typical class description is illustrated in Fig. 5.15. Figure 5.15a shows $f(t)$ defined as lying within an amplitude band over the time interval of interest; Fig. 5.15b illustrates a class of positive inputs characterized by bounds on magnitude, rate of application and duration, and magnitude of prescribed impulse. In discrete form, where $f(t)$ is approximated by a piecewise constant function, the class description amounts to a set

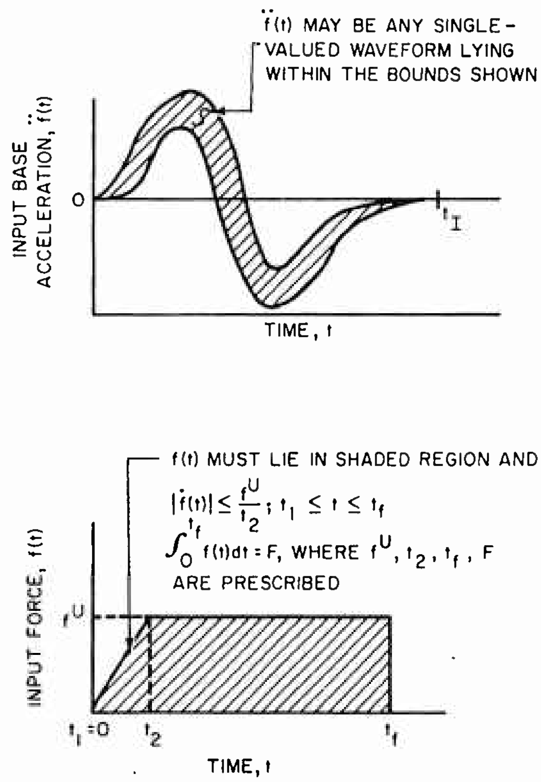


Fig. 5.15. Examples of input class.

of constraint relations on f_i. For example, the band of Fig. 5.15a defines the admissible $f(t)$ as any sequence of numbers f_i satisfying

$$f_i^L \leqslant f_i \leqslant f_i^U; \quad i = 1, 2, \ldots, I, \tag{5.41}$$

where f_i^L, f_i^U are the lower and upper amplitudes of the band prescribed at each time state t_i. The class indicated in Fig. 5.15b would involve other constraining relations among the f_i. In particular, the impulse requirement is

$$\sum_{i=1}^{I} f_i \Delta t = F, \quad F \text{ prescribed}. \tag{5.42}$$

We will consider an example which simplifies notation but does not restrict the generality of the problem formulation. Figure 5.16 illustrates a rectilinear system consisting of an arbitrary arrangement of known elements and a single unprescribed isolator element. This will be termed the general flexible-base isolator system. An input motion of prescribed class description $f(t)$ is imposed on a reference position. No restriction as to linearity of the system is implied.

The equations of motion can be written in the form

$$\dot{\mathrm{x}}(t) = \mathrm{G}(t, \mathrm{x}, u, f) \tag{5.43}$$

$$\mathrm{x}(0) = \mathrm{x}_1 .$$

Note that explicit dependence on the isolator force need only be indicated for the unknown element $u \equiv u(t)$, since the isolator forces for the known elements all involve prescribed relationships among the state variables. The performance index is

$$\psi = \max_s \max_t |h_s(t, \mathrm{x}, u, f)|, \tag{5.44}$$

and the response constraints are

$$C_k^L \leqslant C_k(t, \mathrm{x}, u, f) \leqslant C_k^U; \quad k = 1, 2, \ldots, K. \tag{5.45}$$

Additional constraints which serve to define the class of $f(t)$ are also imposed. All of the functions $\mathrm{G}(\cdot)$, (Eq. (5.43)), $h_s(\cdot)$ (Eq. (5.44)), and $C_k(\cdot)$ (Eq. (5.45)) may be nonlinear. Denote min ψ for an admissible input by ψ^*, i.e.,

$$\psi^* = \min_u \psi .$$

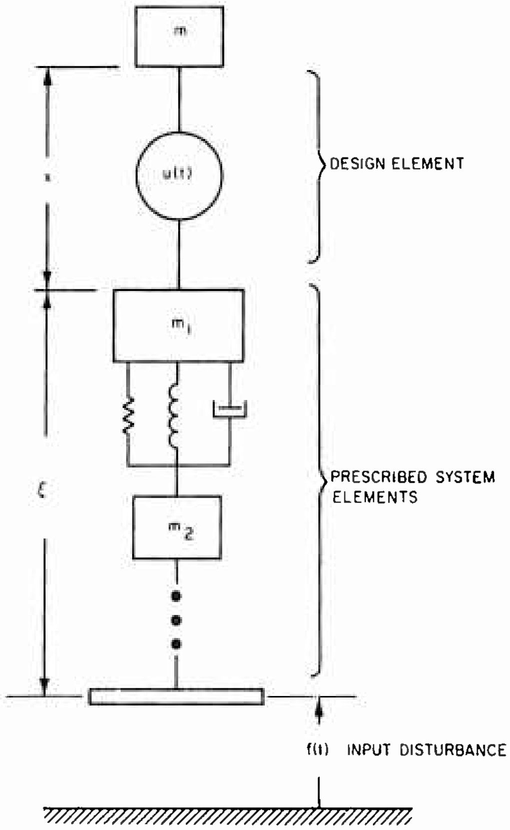


Fig. 5.16. General flexible-base isolator system. The prescribed portion of the system corresponds to any combination of masses and interconnecting elements (e.g., springs, dashpots).

Generally, we would be interested in the largest value of ψ^* for a given class of $f(t)$, the relationship between this ψ^* and the constraints providing an upper bound to the limiting performance characteristic. In the same manner, a lower bound corresponding to the least value of ψ^* for $f(t)$ could be found. Since the solution procedure is formally identical, except for a minimization replacing a maximization, a combined notation will be introduced.

Let the bounding characteristic be denoted by ψ_B^*, where

$$\psi_B^* = \text{opt}\ \psi^* = \underset{f}{\text{opt}}\ \underset{u}{\min}\ \psi. \tag{5.46}$$

The notation *opt* refers to either a minimization (lower bound ψ_{LB}^*) or a maximization (upper bound ψ_{UB}^*). In the process of determining ψ_B^*, an associated

member of the class $f(t)$ is determined. That associated with the lower bound will be termed the *best disturbance* and that associated with the upper bound the *worst disturbance.* Eq. (5.46) admits an LP formulation only for the lower bound, even when all of the functions are linear. Generally, therefore, a dynamic programming solution is required. The complexity of this solution is determined by the size of the problem and not the functional forms; the only solutions reported are for SDF systems [3]. It is doubtful that one would undertake more than a two-degree-of-freedom system at the present time.

The dynamic programming formulation follows exactly the development for an SDF system with fully prescribed input, except that an additional optimization on f is involved. The computational algorithm, analogous to Eq. (5.25), is

$$\phi_{I-i+1}(\mathbf{x}_i) = \underset{f_i}{\text{opt}}\ \underset{u_i}{\min}\ \max\left[h(\mathbf{x}_i, u_i, f_i), \phi_{I-i}(\mathbf{x}_{i+1})\right] \tag{5.47}$$

for

$$i = I-1, I-2, \ldots, 1$$

where

$$h(\cdots) = \max_s |h_s(\cdots)|; \quad s = 1, 2, \ldots, S.$$

The $\mathbf{x}_{i+1}$ are found in terms of the $\mathbf{x}_i$ from the solution of Eq. (5.43). The process starts with

$$\phi_1(\mathbf{x}_I) = \max_t h(\mathbf{x}_I, u_I, f_I); \quad t > t_I.$$

Upon reaching the Ith stage of the process, the desired bounding value is given by

$$\psi_B^* = \phi_I(\mathbf{x}_1),$$

where $\mathbf{x}_1$ refers to the prescribed initial state of the system, Eq. (5.43).

The upper and lower bounds to the limiting performance characteristics are termed the *limiting performance bounds.* A point on the upper bound represents the best performance which can be achieved for the associated constraint level should the system experience the worst disturbance within the specified class. Nothing is said regarding what sort of isolator must be chosen to ensure this optimum performance, or even that some other input within the class might not be more severe on that particular isolator; it simply states what is theoretically possible if an isolator is optimized for the worst disturbance. Computations

made on the basis of the best disturbance ψ_{LB}^{*} are interpreted in a similar fashion. The limiting performance bounds are shown schematically in Fig. 5.17. When additional constraints are involved, the bounds are hypersurfaces. Construction of the upper limiting performance bound is illustrated for an SDF system by Example 7.

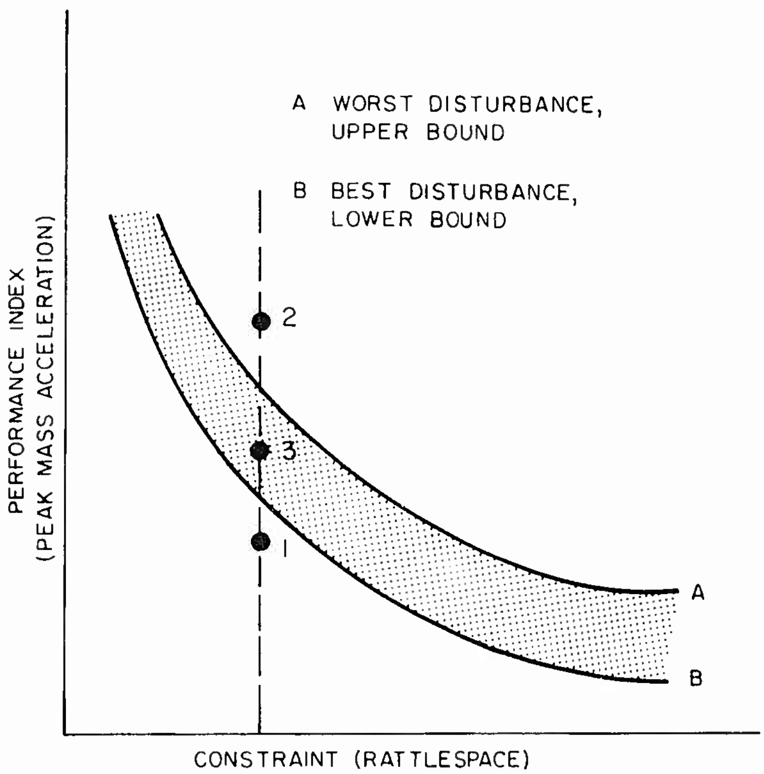


Fig. 5.17. Limiting performance bounds for system with one constraint.

The limiting performance bounds have application for evaluating suggested design criteria or proposed design elements. With reference to Fig. 5.17, a design point lying below the lower bound (Point 1) could not be achieved for any input within the class. On the other hand, a design point lying above the upper bound (Point 2) could be achieved in principle whatever the input. A point lying between the two bounds (Point 3) simply indicates that the desired performance can be achieved for some, but not all, members of the input class.

Example 3

DYNAMIC PROGRAMMING SOLUTION FOR THE TIME-OPTIMAL SYNTHESIS OF AN SDF SYSTEM†

We will consider the SDF system shown in Fig. 4.3. The base structure receives an impulsive loading equivalent to an initial velocity of −20 in./sec. Our problem is to find the least possible rattlespace if the acceleration of the isolated mass is not to exceed 100 in./sec^2. No restriction on the type of isolator is imposed. A unit mass is assumed.

Using the notation of Fig. 5.1, we seek

$$\min_u \psi = \min_u \max_t |x|$$

for which

$$\max_t |\ddot{z}| \leqslant 100 \text{ in./sec}^2.$$

The response quantities x and z are related through the equations of motion which, for unit mass, are

$$\ddot{z} + u(t) = 0$$

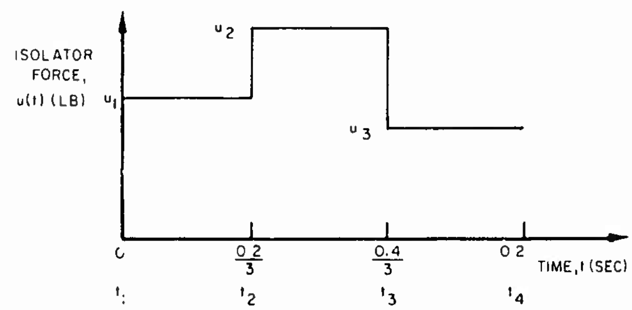
$$z(0) = \dot{z}(0) = 0$$

and through the kinematic conditions

$$z = x + 20t$$

$$\dot{z} = \dot{x} + 20.$$

We will follow the motion for 0.2 sec, during which time the unknown isolator force will be represented in piecewise-constant form as shown. For simplicity, only three subintervals of time are used, i.e., $\Delta t = 0.2/3$.



†It is suggested that this example be studied in conjunction with the general description of the dynamic programming method beginning on p. 42.

In the general notation of Fig. 5.8, $I = 4$ and the problem unknowns are

$$\min \psi$$

$$u_1 = u(t_1) = u(0) = \text{constant for } 0 \leqslant t < 0.2/3 \text{ sec}$$

$$u_2 = u(t_2) = u(0.2/3) = \text{constant for } 0.2/3 \leqslant t < 0.4/3 \text{ sec}$$

$$u_3 = u(t_3) = u(0.4/3) = \text{constant for } 0.4/3 \leqslant t < 0.2 \text{ sec.}$$

From the equations of motion, we see that the acceleration constraint is equivalent to $|u| \leqslant 100$ in./sec^2. Thus, the admissible range for u_i is

$$-100 \text{ lb} \leqslant u_i \leqslant 100 \text{ lb}; \qquad i = 1, 2, 3.$$

The state vector at any time t_i is (see Eq. 5.19)

$$\mathbf{x}_i = \begin{Bmatrix} x_i \\ \dot{x}_i \end{Bmatrix}; \qquad i = 1, 2, 3, 4.$$

To begin the solution we will assume that x_i and $\dot{x}_i$ may take on any values in the range

$$-3 \text{ in.} \leqslant x_i \leqslant 1 \text{ in.}$$

$$-30 \text{ in./sec} \leqslant \dot{x}_i \leqslant 10 \text{ in./sec.}$$

This choice of range for the state variables comes with some experience, but in any event its adequacy is tested during the solution. Again, for simplicity we will select five equally spaced values for the state variables within their respective ranges. In terms of the general notation (p. 43), $N_1 = N_2 = 5$ and there will be $P = (5)(5) = 25$ possible choices for the state vector at each time step considered. These correspond to all combinations of

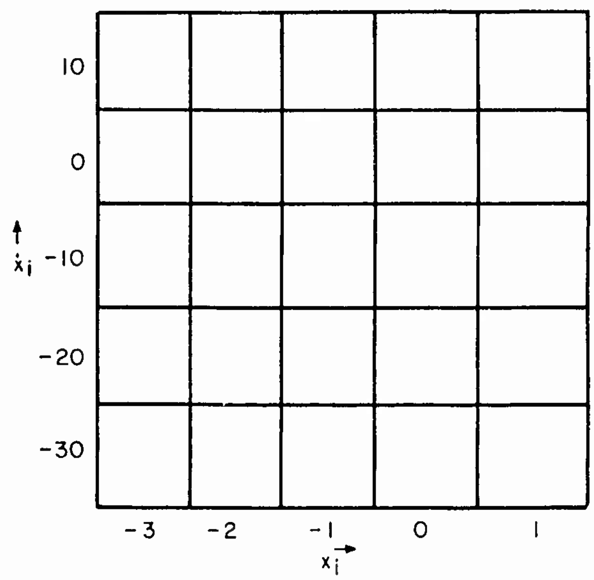
$$x_i = -3, -2, -1, 0, 1 \text{ in.}$$

$$\dot{x}_i = -30, -20, -10, 0, 10 \text{ in./sec.}$$

Rather than use the general recursion relations of Eq. (5.22), we will again derive the dynamic programming algorithm in the course of describing a computational solution. The solution method requires us to find the minimum rattlespace at each time step corresponding to the system being in any of its admissible states at that time. This information will be recorded in a 5 by 5 array, called the ϕ array, as shown schematically below for a typical time t_i.

The columns of the array are labeled for the selected values of x_i, and the rows for the selected values of $\dot{x}_i$. Thus, the intersection of any row and column corresponds to one of the 25 admissible state vectors at the time t_i. Each entry in this array would be the minimum rattlespace requirement (consistent with the acceleration constraint) if the system were to start from the associated state.

For example, if the ϕ array were known at the stage of the computation corresponding to $t_2 = 0.2/3$ sec, then the entry at the intersection of the third row ($\dot{x}_2 = -10$) and second column ($x_2 = -2$) would be the minimum rattlespace required by a system with initial conditions of -2 in., -10 in./sec. Clearly, once the ϕ array is established for $t_1 = 0$,



ϕ - ARRAY

then the entry for $x_1 = 0$, $\dot{x}_1 = -20$ (the actual initial conditions) will be the desired rattle-space, min ψ. The reason for considering all the other states is that the sequence of states corresponding to the motion which produces min ψ is unknown at the outset.

The solution starts at the terminal time $t_4 = 0.2$ sec and proceeds backward in time. The first ϕ array to be considered corresponds to the admissible states $\mathbf{x}_4$ and will be denoted by $\phi_1(\mathbf{x}_4)$ to distinguish between the state of the system and the stage of the solution. Generally, the notation will be $\phi_j(\mathbf{x}_i)$, where the i subscript refers to the time step and the j subscript to the solution step, or stage. There will be as many steps in the solution as there are time increments, so that $j = 1$ when $i = I$ and $j = I$ when $i = 1$. The general relationship between the i and j subscripts is shown in Eq. (5.22).

For our problem, the $\phi_1(\mathbf{x}_4)$ array is filled with zeros, and the computation begins with the $\phi_2(\mathbf{x}_3)$ array. The reason for this will become apparent shortly.

Let us consider how to find a typical entry in the $\phi_2(\mathbf{x}_3)$ array, say that corresponding to the first-row, first-column position, i.e., the state†

$$\mathbf{x}_3 = \left\{-3 \text{ in.}, 10 \text{ in./sec}\right\}.$$

At this time ($t_3 = 0.4/3$ sec) the mass can experience any value of the isolator force u_3 between ± 100 lb. A particular value of u_3 will cause the system to be in a new state at a time Δt later, which can be found from the solution to the equations of motion. The state transformation results from the solution to the equations of motion, and for unit mass is

†The state vector is written in its transpose form for convenience.

$$x_{i+1} = x_i + \dot{x}_i \Delta t - \frac{1}{2} u_3 (\Delta t)^2$$

$$\dot{x}_{i+1} = \dot{x}_i - u_3 \Delta t.$$

For example, if we select $u_3 = -100$ lb, then the state $\mathbf{x}_3 = \{-3 \text{ in. } 10 \text{ in./sec}\}$ is transformed to the state $\mathbf{x}_4 = \{-2.1\text{i in. } -3.33 \text{ in./sec}\}$. We observe from this result that the minimum rattlespace is at least 2.11 in. whether or not a larger value for subsequent motion is required, as indicated by the entry foı the -3.33 in./sec row and -2.11 in. column of the $\phi_1(\mathbf{x}_4)$ array. These are not actual column labels, and it is necessary to interpolate in the array. However, since the $\phi_1(\mathbf{x}_4)$ array is filled entirely with zeros, we conclude that a system starting from the state $\mathbf{x}_3 = \{-3 \text{ in., } 10 \text{ in./sec}\}$ and experiencing an isolator force of $u_3 = -100$ lb will require a rattlespace of 2.11 in.

Of course, there are other values of u_3 to be considered, and this calculation must be repeated for representative values of u_3 within the range ± 100 lb. The minimum value of the rattlespace so determined is then entered into the first row, first column of the $\phi_2(\mathbf{x}_3)$ array. The value of u_3 which causes the minimum rattlespace u_3^* may be recorded also. It should be clear that this computation may be expressed symbolically as

$$\phi_2(\mathbf{x}_3) = \min_{u_3} \max \left[|x_4(\mathbf{x}_3, u_3)|, \ \phi_1(\mathbf{x}_4) \right].$$

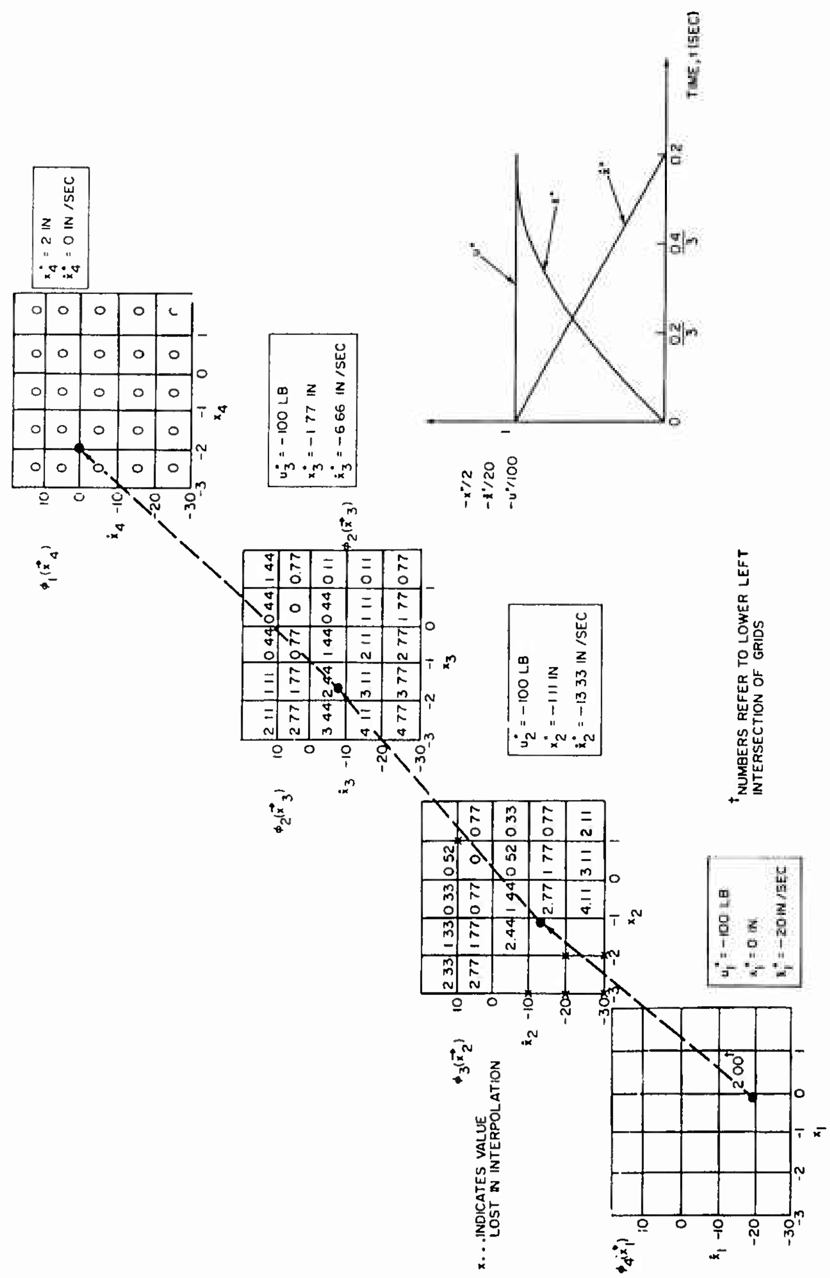
The dependence of the rattlespace on the choice of the state variables at time t_3 is indicated explicitly, as is the indication that the computation is to be carried out for each of the admissible state vectors. It also should be evident that the choice of zeros for $\phi_1(\mathbf{x}_4)$ is not a limitation, since the largest rattlespace requirement should have been established prior to the end of the time interval; otherwise, the interval would not be of sufficient length.

The results for the $\phi_2(\mathbf{x}_3)$, $\phi_3(\mathbf{x}_2)$, and $\phi_4(\mathbf{x}_1)$ arrays are shown in the example figure. Ten values of u_3 were selected for the computation. Only the known initial state $\mathbf{x} = \{0.0 \text{ in. } -20 \text{ in./sec}\}$ need be considered in the $\phi_4(\mathbf{x}_1)$ array, and the entry shown is the desired solution for minimum rattlespace,

$$\min_{u} \psi = \min_{u} \max_{t} |x| = 2 \text{ in.}$$

Despite the crudity of this example, the result is exact, as a consequence of the initial velocity input and the fact that the optimum isolator force is of the bang-bang type, which is correctly represented by the piecewise constant approximation to $u(t)$.

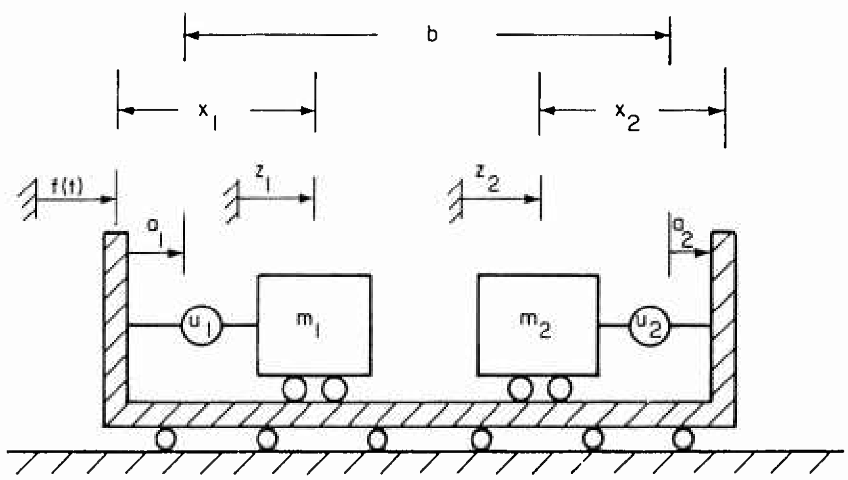
If desired, a separate calculation, proceeding forward in time, is required to establish the optimum isolator force $u^*(t)$ and the trajectories of the state variables during the 0.2-sec time interval. In the course of computing $\phi_4(\mathbf{x}_1)$ for the initial state $\mathbf{x}_1 = \{0 \text{ in. } -20 \text{ in./sec}\}$, the first increment of force u_1^* is found to be -100 lb. This causes the system to be in the state $\mathbf{x}_2^* = \{-1.11 \text{ in. } -13.33 \text{ in./sec}\}$ at $t_2 = 0.2/3$ sec. Either a recalculation of min u_2 for $\phi(\mathbf{x}_2^*)$ or interpolation among the min u_2 associated with the calculation of $\phi_3(\mathbf{x}_2)$, if this information was retained, yields $u_2^* = -100$ lb. This force increment, in turn, causes the system to be in state $\mathbf{x}_3^* = \{-1.77 \text{ in., } -6.66 \text{ in./sec}\}$ at $t_3 = 0.4/3$ sec. Similarly, $\phi(\mathbf{x}_3^*)$ is found to occur for $u_3^* = -100$ lb. This last increment of the isolator force causes the system to be in the state $\mathbf{x}_4^* = \{-2.00 \text{ in., } 0 \text{ in./sec}\}$, indicating that the system comes to rest at the end of the 0.2-sec time interval. These trajectories are plotted in the accompanying figure, along with an indication of the path through the ϕ arrays. The row and column intersections are depicted as points, with the values of ϕ indicated, in order to emphasize the interpolation requirements.


TIME, t (SEC)
†NUMBERS REFER TO LOWER LEFT INTERSECTION OF GRIDS
x . . . INDICATES VALUE LOST IN INTERPOLATION

Example 4

THE PROXIMITY PROBLEM

We will consider two independently isolated packages m_1 and m_2, coupled only by virtue of the possibility of their contact. The mounting structure undergoes the prescribed motion $f(t)$.



The equations of motion are

$$\left.\begin{aligned} m_1\ddot{z}_1 + u_1 &= 0 \\ m_2\ddot{z}_2 + u_2 &= 0 \\ z_1 &= f + [x_1 - x_1(0)] \\ z_2 &= f - [x_2 - x_2(0)] \end{aligned}\right\} \qquad \begin{aligned} m_1\ddot{x}_1 &= -u_1 - \ddot{f} \\ m_2\ddot{x}_2 &= +u_2 - \ddot{f} \end{aligned}$$

$$z_1(0) = \dot{z}_1(0) = z_2(0) = \dot{z}_2(0) = 0.$$

While this appears to suggest a completely symmetrical arrangement, recall that the u_j may involve prescribed elements which differ (Fig. 5.2).

The performance index is chosen to be the greatest of the peak accelerations experienced by either mass. Thus,

$$\psi = \max[\max_t|\ddot{z}_1|, \max_t|\ddot{z}_2|]\,; \qquad \ddot{z}_j = \frac{u_j}{m_j}\,.$$

As constraints, we require that neither mass approach within a prescribed distance of the walls and that neither one contact the other. These conditions are expressed as

$$x_1 > a_1$$

$$x_2 > a_2$$

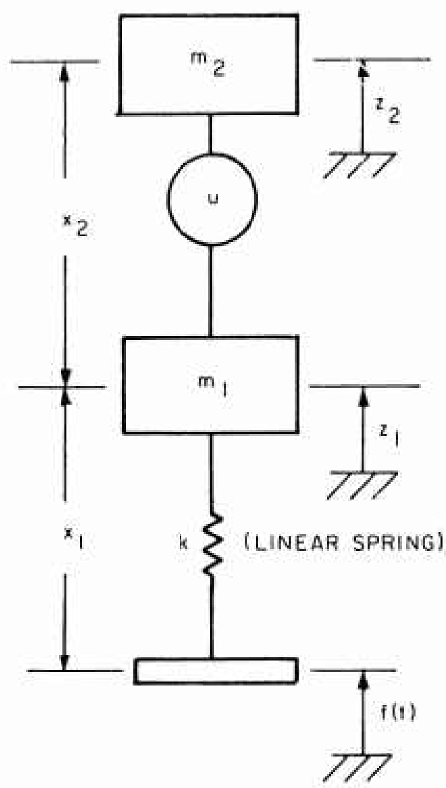
$$x_1 + x_2 < b.$$

The relationships between x_1 and u_1, and x_2 and u_2 are most simply found by integrating the equations of motion.

The optimization solution yields the isolator forces $u_1(t)$ and $u_2(t)$ which, for given clearances a_1, a_2, and b, and initial positions of the masses, minimize the larger of the peak accelerations experienced by the two masses. The limiting performance characteristics result from repeated solutions for various values of $x_1(0), x_2(0), a_1, a_2$, and b. These characteristics provide the minimum possible mass accelerations corresponding to given initial mass positions and housing clearances. Alternatively, for a desired peak acceleration level the minimum clearance between the masses is provided.

Example 5
FLEXIBLE-BASE MODEL

We will consider an improvement to the rigid-base, rigid-package, SDF system.



The simple elastic model of the base structure shown reproduces a single, undamped response mode; the package is represented by the rigid mass m_2. The peak acceleration of m_2 is selected as the performance index. The rattlespace required by the package and the deformation of the base are constrained. Thus, we seek the $u(t)$ which satisfies

$$\max_t |x_1| \leqslant D_1$$

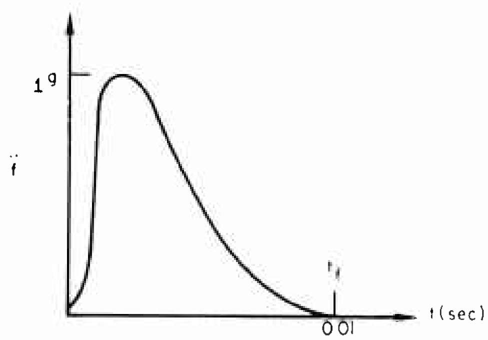
$$\max_t |x_2| \leqslant D_2$$

and minimizes

$$\psi = \max_t |\ddot{z}_2|.$$

The response variables required by the LP formulation are easily established by using the equations of motion.

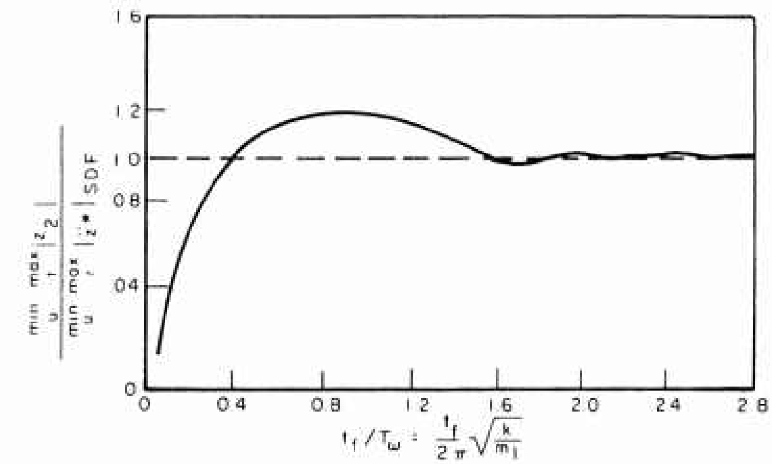
Numerical solutions were computed for a range of base frequencies and an input acceleration pulse of the form shown below.



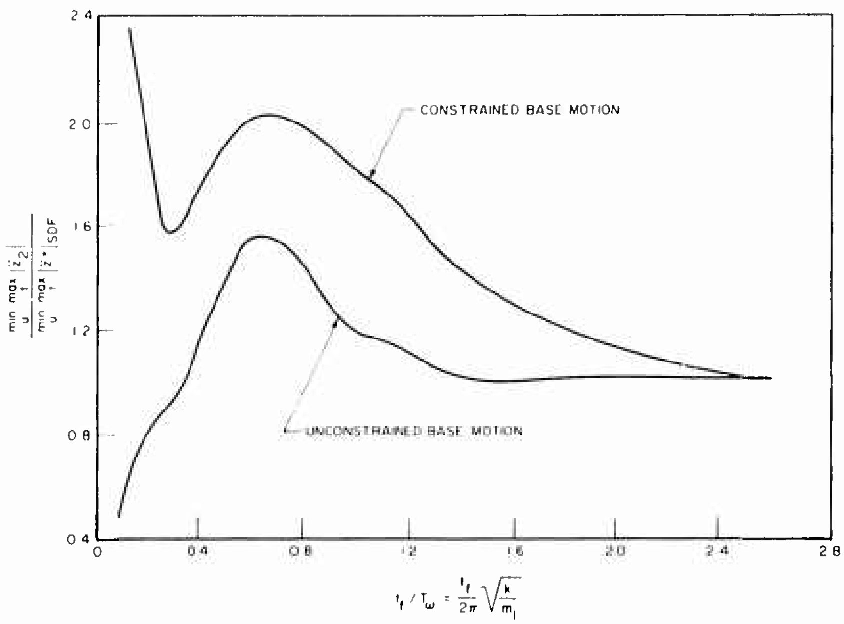
The rattlespace bound was set at 30 percent of the base displacement occurring at the duration of the pulse; the deformation of the base was unconstrained. The minimum peak acceleration, expressed in terms of the rigid-base problem, is plotted against the ratio of the pulse duration to the period of the base structure for equal masses ($m_1/m_2 = 1$) in the figure at the top of the next page. Here, T_ω is the period of motion of the base structure.

Minimum ψ for the rigid base is 36 percent of the peak base acceleration. The limiting isolator performance becomes effectively that of the rigid-base system for base periods less than about 60 percent of the pulse duration. This result depends on the fact that the relative displacement was unconstrained. For a range of periods in excess of the pulse duration, minimum peak transmitted accelerations exceed optimum conditions for a rigid base by as much as 20 percent. Since no restrictions were placed on base displacement, the optimum transmitted acceleration approaches zero as the base flexibility increases.

Similar results are shown in the following figure for a base mass twice that of the package ($m_1/m_2 = 2$). The lower of the two curves is to be compared directly with the results of the previous figure. For the relatively less massive package, the effectiveness of the optimum isolator is further reduced for some base frequencies. Also, there appears to be an enhanced tuning effect; i.e., a lesser range of frequencies over which the minimum peak transmitted



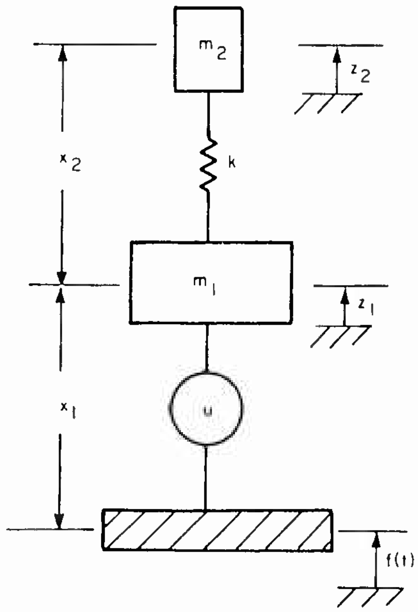
acceleration significantly exceeds the rigid-base results. The upper curve is also for a mass ratio of 2, but corresponds to a displacement constraint on the base structure of 110 percent of the base displacement in the absence of the package mass ($m_2 = 0$). This is in line with the requirement that the addition of the package and its isolator not significantly increase the stresses in the support structure. As expected, the influence of the constraint raises the minimum transmitted acceleration for all base frequencies, and most noticeably for increasing base flexibility.



Example 6

FLEXIBLE-PACKAGE MODEL

A two-mass model of the package structure provides for a single, undamped response mode of a subcomponent; the base is assumed rigid.



The performance index is taken to be the maximum acceleration of the package; i.e.,

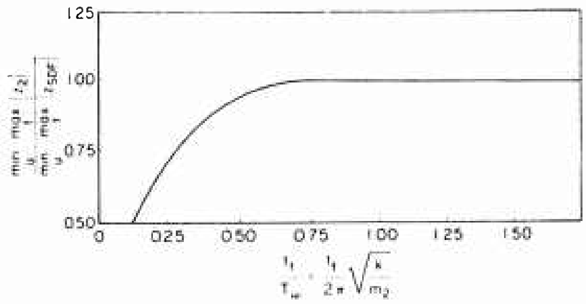
$$\psi = \max_t |\ddot{z}_2|.$$

Constraints are imposed on the isolator force and rattlespace; i.e.,

$$\max_t |u(t)| \leqslant F$$

$$\max_t |x_1(t)| \leqslant D.$$

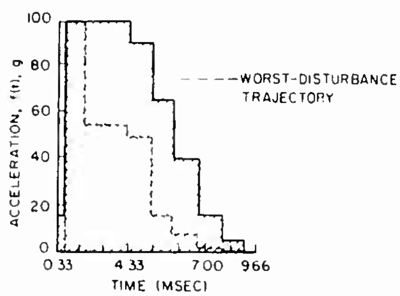
Numerical results were obtained for the same input pulse as in Example 5, a rattlespace bound equal to 30 percent of the base displacement occurring at the duration of the pulse, and a mass ratio $m_1/m_2 = 100$. The value of F was sufficiently large so as not to be an active constraint. The following figure shows the minimum peak acceleration transmitted to m_2, normalized to the corresponding SDF case, plotted against the ratio of pulse duration to the period of the package mode. For periods less than about twice the pulse duration, the limiting performance is essentially that of the rigid-package model. For greater periods (increasingly flexible package) the minimum peak acceleration approaches zero.



Example 7

LIMITING PERFORMANCE BOUNDS FOR AN SDF SYSTEM

We will illustrate the manner of determining a point on the upper limiting performance bound and the associated worst disturbance for an SDF system subject to the input class shown.



The input acceleration must lie within the shaded region and have a net area of 147.8 ips. The input is described as a bounded acceleration pulse of specified terminal velocity V_f. The peak transmitted acceleration of the mass is taken as the performance index with a constraint on relative displacement. Piecewise constant approximations are used for both $u(t)$ and $\ddot{f}(t)$. The terminal constraint on the input class has the effect of increasing by one the number of state variables and, hence, the dimension of the dynamic programming solution.

The solution to the system equations, in a form required by Eq. (5.47), is

$$\mathbf{x}_{i+1} = \begin{Bmatrix} x_{i+1} \\ \\ \dot{x}_{i+1} \end{Bmatrix} = \begin{Bmatrix} x_i + \dot{x}_i \Delta t - \dfrac{1}{2}\left(\ddot{f}_i + \dfrac{u_i}{m}\right)(\Delta t)^2 \\ \\ \dot{x}_i + \left(\ddot{f}_i + \dfrac{u_i}{m}\right)\Delta t \end{Bmatrix}$$

with $x_1 = \dot{x}_1 = 0$ and where $\ddot{f}_i$ has the units of acceleration.

The performance index is $\psi = \max_i |\ddot{z}_i| = \max_i |u_i/m|$, and the response constraint is $-D \leqslant x_i \leqslant D$.

The input class is defined by

$$0 \leqslant \ddot{f}_i \leqslant \ddot{f}_i^U$$

$$\sum_{i=1}^{I} \ddot{f}_i \Delta t = V_f,$$

where $\ddot{f}_i^U$ is the time-varying upper bound shown in the figure.

To handle the terminal velocity constraint, we define an additional state variable r according to

$$r_{i+1} = r_i + \ddot{f}_i \Delta t.$$

At the start of the computational process, $r_I = 0$; at its conclusion, $r_1 = V_f$. Thus, r_i records the cumulative velocity, and the complete state vector is

$$\mathbf{x}_i = \begin{Bmatrix} x_i \\ \dot{x}_i \\ r_i \end{Bmatrix}; \qquad \mathbf{x}_1 = \begin{Bmatrix} 0 \\ 0 \\ V_f \end{Bmatrix},$$

where the first two components of $\mathbf{x}_i$ correspond to those above.

For the upper bound (worst-disturbance) case, Eq. (5.47) takes the form

$$\phi_{I-i+1}(\mathbf{x}_i) = \max_{f_i} \min_{u_i} \max \left[\left| \frac{u_i}{m} \right|, \phi_{I-i}(\mathbf{x}_{i+1}) \right]$$

for

$$i = I-1, I-2, \ldots, 1.$$

A solution was obtained for

$$m = 1 \text{ lb sec}^2/\text{in.}$$

$$V_f = 148 \text{ in./sec}$$

$$D = 0.25 \text{ in.}$$

The upper bound to the minimum transmitted acceleration is found to be

$$\psi^*_{UB} = \phi_I(0) = 19{,}300 \text{ in./sec}^2.$$

The associated worst disturbance is shown by the dashed line within the bounded input class.

If this procedure is carried out for a range of values of the constraint, the upper limiting performance bound (associated with the worst disturbance) is constructed.

Chapter 6

OPTIMUM DESIGN SYNTHESIS OF SHOCK ISOLATION SYSTEMS

In the general formulation of the optimum design-parameter problem considered in Chapter 4, the configuration of each of the isolator elements is presumed known, but a number of parameters (e.g., spring and damping rates) are unspecified as to numerical value. The synthesis problem is to select these design parameters so that the performance index is minimized without violating the constraints. Any method that seeks to do this by continuously satisfying the constraints and progressively minimizing the performance index is termed a *direct synthesis* method. By contrast, an *indirect synthesis* method is one that selects the design parameters on the basis of approximating the isolator response trajectories that produce the limiting performance.

It should be kept in mind that while the result of either method may be termed an optimum design, it is optimum only with respect to the type of isolator being considered. Whether or not some other type of isolator may yield better performance cannot be known without repeating the synthesis procedure for that isolator. Thus, *how optimum is optimum* only can be found by comparing the local optimum with the limiting performance determined by the methods of Chapter 5.

In addition to presenting both methods of synthesis, this chapter includes a discussion of the influence of uncertainty of the input details on the optimum performance characteristics. The latter material also is applicable to the limiting performance characteristics discussed in Chapter 5.

6.1 Direct Synthesis

6.1.1 Completely Described Environment

Analytical Techniques

Direct synthesis is a problem of constrained minimization for which analytical methods are practical only when the number of unknown parameters is small or when the performance index is of a particularly convenient form. We will consider a simple system that illustrates a rather straightforward approach and points up the difficulties encountered in extending it to more complicated systems.

Figure 6.1 shows a linear spring-dashpot isolator subject to an impulse loading of its base equivalent to the initial velocity V. The peak acceleration of the mass is selected as the performance index, and the rattlespace is constrained. In addition, the spring rate k is to be nonnegative and the system overdamped. Thus, the performance index is

$$\psi = \max_t |\ddot{z}|. \tag{6.1}$$

The constraints are

$$\begin{aligned} \max_t |x| &\leqslant D \\ k &\geqslant 0 \\ c &\geqslant 2\sqrt{km}\,, \end{aligned} \tag{6.2}$$

and the equations of motion are†

$$m\ddot{z} + c\dot{x} + kx = 0 \tag{6.3}$$

with

$$x = z - f = z - Vt$$

and

$$z(0) = \dot{z}(0) = 0.$$

Our problem is to select the design parameters k and c so that the constraints of Eq. (6.2) are satisfied and the performance index of Eq. (6.1) minimized. We do this by first finding the expression for ψ in terms of the design parameters. The solution to Eq. (6.3) is

$$x(t) = -\frac{V}{2\Omega}\left[e^{\gamma_1 t} - e^{\gamma_2 t}\right], \tag{6.4}$$

†Distributed mass effect in the spring (surging) is neglected, which might not be adequate in a practical situation involving impact loads.

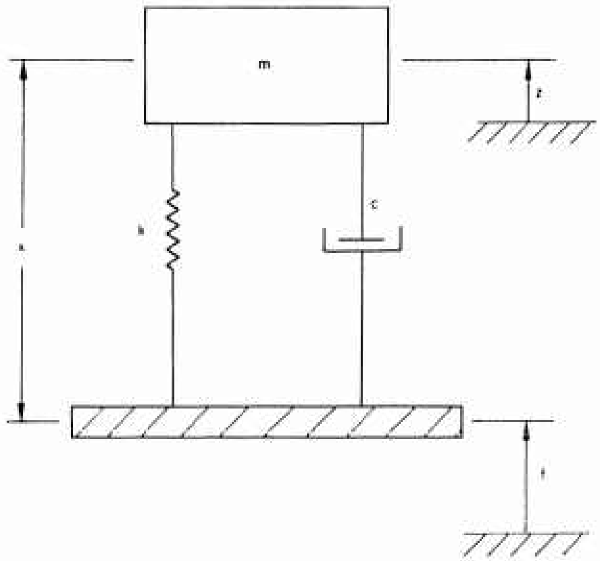


Fig. 6.1. Linear spring-dashpot isolator.

where

$$\gamma_1 = -\frac{c}{2m} + \Omega$$

$$\gamma_2 = -\frac{c}{2m} - \Omega$$

$$\Omega = \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}}\,.$$

Note that requiring the dashpot to be overdamped implies that Ω will be real.

The maximum relative displacement has the value

$$\max_t |x(t)| = x(t_m) = \frac{V}{\gamma_2}\left(\frac{\gamma_2}{\gamma_1}\right)^{\gamma_1/2\Omega}, \tag{6.5}$$

where

$$t_m = \frac{1}{2\Omega}\, ln\left(\frac{\gamma_2}{\gamma_1}\right).$$

The maximum acceleration $\ddot{z}$ is

$$\max_t |\ddot{z}| = \ddot{z}(0) = \frac{Vc}{m}\,.$$

The constrained minimization problem now can be written in the following explicit form: Find k and c such that $\psi = (Vc)/m$ is a minimum and

$$\frac{V}{\gamma_2}\left(\frac{\gamma_2}{\gamma_1}\right)^{\gamma_1/2\Omega} \leqslant D$$

$$k \geqslant 0$$

$$c \geqslant 2\sqrt{km}$$

with V, m, and D prescribed constants.

We see that despite the linear form of the system, the minimization problem for k and c is highly nonlinear, as generally will be the case. The min ψ is found (e.g., graphically) to occur for the condition of critical damping ($\Omega = 0$), which is a singular solution of Eq. (6.3). There results

$$\psi^* = A = \frac{2V^2}{eD}\ ; \qquad e = 2.718\ldots$$

$$c^* = \frac{2Vm}{eD} \tag{6.6}$$

$$k^* = \frac{c^{*2}}{4m} = \frac{V^2 m}{e^2 D^2}\ ,$$

where the optimum values of the design parameters are again designated by asterisks.

This result is compared in Table 6.1 with the limiting performance, Eq. (5.10), and the situations in which either the spring or dashpot is absent:

Table 6.1. Optimum Parameters

Parameters†	Limiting Performance (Eq. 5.10)	Optimum Spring Dashpot (Eq. 6.6)	Optimum Spring	Optimum Dashpot
A	1/2	$2/e$	1	1
k^*	—	$1/e^2$	1	—
c^*	—	$2/e$	—	1

† $A = \text{coef}_A \dfrac{V^2}{D}$; $k^* = \text{coef}_{k^*} \dfrac{V^2 m}{D}$; $c^* = \text{coef}_{c^*} \dfrac{Vm}{D}$.

Coef_A, coef_{k^*}, coef_{c^*} are given by the A, k^*, c^* rows, respectively.

The "optimum" spring and dashpot cases do not really represent a problem of optimization, since if k or c is selected so that $\max|x| = D$, $\max|\ddot{z}|$ is determined.

We observe that, for any prescribed input velocity and rattlespace, the best spring-dashpot isolator will transmit a peak acceleration about one-third greater than the best possible isolator, whereas eliminating either the spring or dashpot doubles the acceleration transmission relative to the limiting performance. An appreciation of the sensitivity of the optimum design to the selected configuration in this instance can be gained by noting how the performance would change if the dashpot were eliminated after the spring had been selected on the basis of a spring-dashpot isolator. Here we see that, for the same base velocity, the peak acceleration would be halved, but the rattlespace would be exceeded almost threefold (by the factor e). These results, of course, apply only for the impulse loading, but the trends are believed similar for other inputs.

The control theory literature abounds with methods suitable for direct optimum synthesis of very simple systems on the basis of an integral type of performance index, usually the so-called integral square or quadratic criterion. Problems involving this type of performance index can be handled with calculus-of-variations techniques, and solutions have been obtained for SDF shock isolation systems. While it is conceivable that some equipment can withstand reasonably large responses of short duration so long as the average response is not excessive, peak response indices generally are considered more applicable to isolation system design. Applicable techniques include Pontryagin's maximum principle [1, 5], an analytical version of dynamic programming [23], and the minimization of auxiliary effort [24]. Both the optimum system configuration, which as a result of the quadratic performance index is linear [1], and the optimum design parameters can be found. However, there is little indication that these methods can be applied to realistic models of complex isolation systems or extended to encompass performance criteria more appropriate to the shock isolation problem.

Computational Techniques

Computational techniques for optimizing a particular isolation system design, in most instances, are the only reasonable approaches for direct synthesis. Since many comprehensive reviews of numerical minimization techniques are available [25-33], a discussion of the details of the methods and their relative merits is not included. Rather, we will review certain common features of those methods and cite a few results. As will be shown, the practicality of any of these methods is limited by the size of the system and the number of design parameters.

All numerical methods of direct synthesis progress toward the desired minimum in an iterative fashion. At any stage in the process, they select a trial set of the design parameters, solve the equations of motion, and then test to see if the response constraints are satisfied. If the constraints are satisfied, the performance index is evaluated and compared to the minimum value thus far obtained; the current minimum and associated design parameters are retained and another set

of design parameters is selected. If the constraints are not satisfied, this trial set of design parameters is rejected and another set selected. The various minimization techniques differ principally in their manner (a) of deriving the next trial set of parameters from the results of preceding trials and (b) of verifying the constraints.

A different approach to the latter aspect distinguishes the so-called penalty function methods, which seek to replace the original constrained minimization problem by a sequence of unconstrained minimizations. A new performance index is constructed that reduces to the original performance index when the constraints are satisfied and weights (i.e., penalizes) the index when they are not. The Fiacco-McCormick method is one of the most popular and powerful of the penalty-function techniques [34].

The performance index may be thought of as representing a surface in design-parameter space (a hypersurface, if more than two design parameters are involved), with the constraint functions serving to restrict the admissible region of the space. The optimum design is the minimum altitude of the response hypersurface within the admissible region of the design space. Viewed in this manner, direct synthesis amounts to a search procedure for exploring the topology of this hypersurface, where the description and positions of the boundaries of the hypersurface must be found by solution of the system dynamics. Schmit [20] provides an excellent discussion of the problem from a geometric point of view.

Regardless of the computation algorithms involved, all of the numerical search methods possess the following features:

- The system dynamics (i.e., equations of motion) must be solved for each trial set of design parameters.
- The computational burden increases with the number of degrees of freedom of the overall system, the number of design parameters, and the number of constraints involving the state variables.
- Convergence of the search procedure to the minimum is not always guaranteed. When the procedure does converge, it characteristically does so in a relatively great number of steps (hundreds), and it is seldom known whether convergence is to a relative or a global minimum.
- The procedure is not additionally complicated by the functional form of the performance index or constraints except as these affect convergence.

There is considerable activity at present in the development of larger and more efficient codes for solving problems of constrained minimization (e.g., Ref. 30). Of particular interest are the methods presently being investigated to avoid repeated solutions of the system dynamics at each iteration of the search procedure [35-37].

While most current developments involve digital computation, some isolation system synthesis has been performed by analog means [5, 17]. At one time, of course, variation-of-parameters studies were thought to be the exclusive preserve

of the analog computers and, indeed, they possess desirable features. However, a discussion of analog computer simulation of isolation systems and current developments in hybrid computation is beyond the scope of this monograph.

Reference 38, an extension of the study described in Ref. 39, deals with the optimum synthesis of an SDF isolation system consisting of a bilinear spring and time-dependent damping. The problem formulation and representative results are presented in Example 8. Inasmuch as the optimum values of some of the design parameters were found to depend on the starting values for the search procedure, there is some question as to whether an absolute minimum for ψ was obtained. This is typical of what one must expect in the solution to a multiple-parameter system, where convergence of the search procedure to even a relative minimum cannot be guaranteed.

6.1.2 Incompletely Described Environment

When the input is described as a finite family of waveforms, each of which may occur with equal probability, the system response must be examined for all waveforms for each trial set of design parameters to establish admissible designs. It may well be that the input which causes the constraints to become operative differs from that which provides the minimum performance index. Optimization of the shock isolation system of Example 8 was performed by direct synthesis in Ref. 38 for a multiple-input situation.

The method of direct synthesis is complicated considerably when applied to a class of inputs, since a worst-disturbance analysis must first be carried out for each trial set of design parameters to ensure that no constraint is violated. The worst disturbance is that member of the input class for which a designated response quantity takes a maximum value; this method of analysis is described in Section 6.3.

Recall that the synthesis problem requires us to find the design parameters a_{jr} (R parameters for each of J isolators) that minimize the performance index while satisfying all of the imposed constraints. The constraints are of two types: (a) design-parameter constraints, i.e., conditions on the a_{jr}, and (b) response constraints, i.e., conditions on the state variables. The direct-synthesis method starts by selecting a trial set of the a_{jr} consistent with the constraints on these parameters. It is next determined whether for these a_{jr} the system satisfies the response constraints. A worst-disturbance analysis† is carried out for each of the response constraint functions to ensure that there is no member of the input class that would cause any response constraint to be violated for the particular choice of a_{jr}. The form of the system dynamics and constraint functions will determine whether the dynamic programming or the more efficient LP solution can be used for this analysis.

†A best-disturbance analysis would be conducted for a constraint which required a response function to exceed a given value.

If all of the constraints are satisfied for the extreme disturbance among the class of inputs, then the trial set of design parameters is admissible. If not, a new set must be chosen and the process repeated. It may result that the response constraints are incompatible with the class of inputs or with the restrictions on the design parameters. In such an event, one or the other must be relaxed or else other candidate isolators considered.

Assuming that an admissible set of a_{jr} has been found, the next step is to calculate the performance index ψ_B^* associated with the worst disturbance. This is the largest the performance index need be for any input within the prescribed class, but it may be reduced for some other choice of the a_{jr}. Therefore, ψ_B^* is viewed as the objective function of a mathematical programming problem in which a search is made for a new set of the a_{jr} providing a lesser value of ψ_B^*. If a lesser value is found, the new a_{jr} must be tested for admissibility as before and the above procedure repeated in its entirety. The process concludes when no further improvement in ψ_B^* is possible for admissible a_{jr}. In general, the extreme disturbances leading to the optimum performance index will differ from the extreme disturbances that generate extreme values of the response constraint functions. This procedure is summarized in the flow chart of Fig. 6.2 and applied to an SDF isolator system in Example 9. This example is taken from Ref. 3, which appears to be the only published solution of direct synthesis for an input class description.

6.2 Indirect Synthesis

6.2.1 Completely Described Environment

The method of indirect synthesis seeks to determine the optimum design parameters for a selected isolator configuration on the basis that the isolator force-time variation (or some other function of the state variables) approximates the optimum isolator time characteristics. The basic assumption is that if a selected isolator responds sufficiently like the ideal isolator, then the constraints will be satisfied and the performance index minimized. The indirect synthesis approach requires that the time-optimal response be obtained according to the methods of Chapter 5 first; however, solving the constrained minimization problem of direct synthesis usually more than makes up for this effort.

Indirect synthesis is a new method, and it has not been demonstrated that an optimum design must necessarily result by approximating the theoretical optimum response. However, the potential computational advantages offered by the method are great and the results reported so far are encouraging. We describe a general approach and several methods of approximating the optimum response.

General Approach

We consider a shock isolation system comprising J isolator elements, each of which is to be designed optimally. Let the state vector which describes the

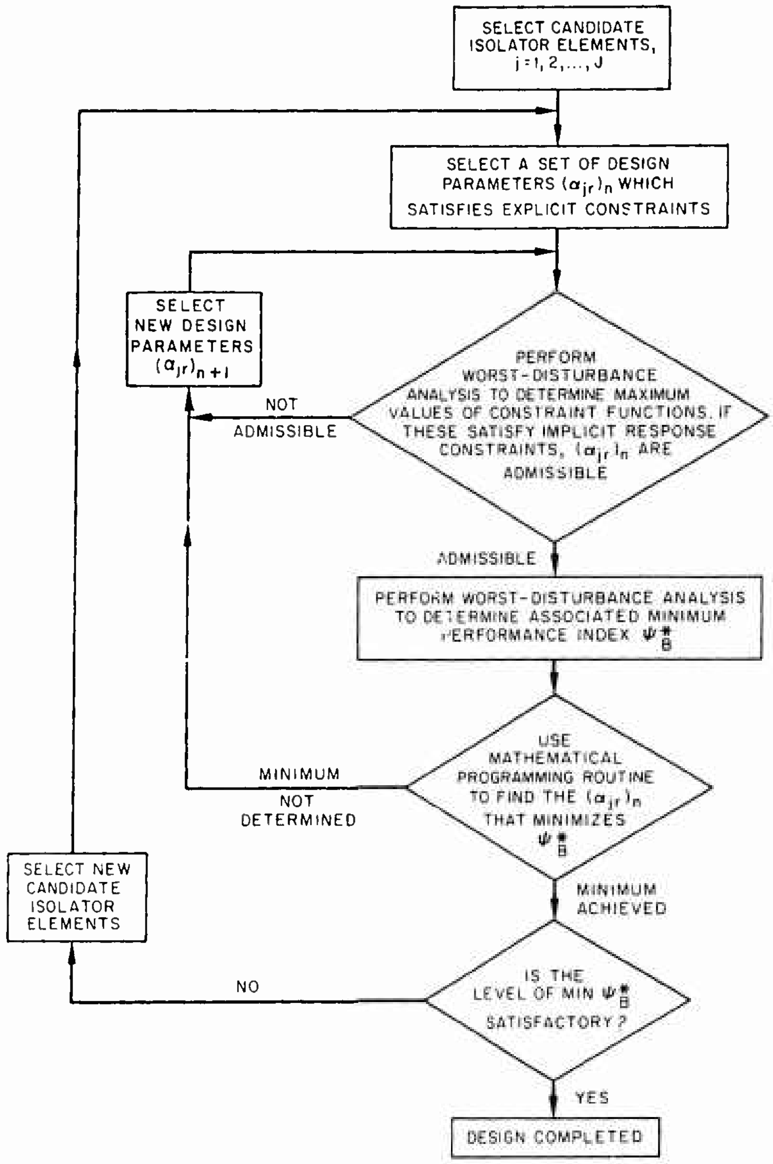


Fig. 6.2. Direct synthesis procedure for an input class description.

relative motion of the isolator terminals be denoted by $\mathbf{x}_j$ and the net force across the isolator by u_j ($j = 1, 2, \ldots, J$). We assume that the configuration of each of the isolators has been selected, so that each u_j can be written as an explicit function of $\mathbf{x}_j$, the time t, and certain design parameters a_{jr} ($r = 1, 2, \ldots, R$); i.e.,

$$u_j = u_j(x_j, \dot{x}_j; a_{jr}). \tag{6.7}$$

The design problem, once again, is to select the a_{jr} for each isolator so that the performance index is minimized and all of the imposed constraints satisfied. In the method of indirect synthesis we further assume that the limiting performance characteristic has been determined according to the methods of Chapter 5. That is to say, the quantities $u_j^*(t), x_j^*(t)$, and $\dot{x}_j^*(t)$ are known. These are referred to as the *time-optimal response functions.* We will attempt to determine the a_{jr} by requiring the u_j to approximate the u_j^* without directly imposing the response constraints or seeking to minimize the performance index. Hence, we call this an *indirect synthesis* method.

Table 6.2 lists a number of rectilinear isolator-element configurations and their associated force functions u_j. We see that, in general, the u_j of Eq. (6.7) can be represented by a nonlinear ordinary differential equation of the form

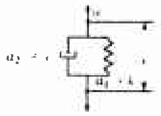
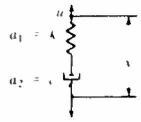
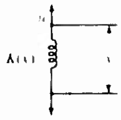
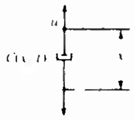
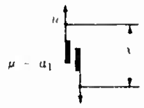
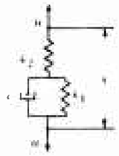
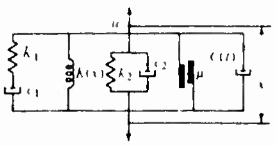
$$U_j(u_j^p, u_j^{p-1}, \ldots, u_j;\; x_j^q, x_j^{q-1}, \ldots, x_j;\; a_{jr};\; t) = 0, \tag{6.8}$$

subject to appropriate initial conditions (superscripts denote time derivatives).

Our approach will be to replace appropriate arguments of Eq. (6.8) by their time-optimal values (denoted by asterisks) and then determine the a_{jr} so that the equation is approximately satisfied. This is related closely to a more general problem known as *system identification* which is gaining increasing interest in a variety of applications. In the present context, system identification deals with finding the unknown parameters of a system of differential equations so that a particular solution best approximates empirical data. Here, the data are supplied by the time-optimal functions.

Regardless of the system identification approach employed, the more time-optimal data available and the smoother these data, the easier it should be to identify the optimum design parameters. One method of accomplishing this is to select the discretization schemes for the isolator force u_j used in the time-optimal solution judiciously. Rather than employing simple piecewise constant or linear representations, approximations that ensure continuity or even differentiability of u_j may be preferable. These later representations of u_j can be handled by the mathematical programming formulation, although the value of the minimized performance index may well be higher than that obtained through the less restrictive piecewise constant or linear discretization. Not only would the optimum isolator force u_j^* be smoother, and hence more desirable from a curve-fitting standpoint, but with the proper discretization the first and perhaps higher derivatives of u_j^* would be available for use in Eq. (6.8). The isolator force can also be smoothed by invoking constraints such as restrictions on rise times.

Table 6.2. Representative Isolator Elements

Name	Configuration	Isolator Force—Relative Displacement Relation
Voigt Element		$u = a_1 x + a_2 \dot{x}$
Maxwell Element		$\dot{u} + \frac{a_1}{a_2} u = -a_1 x$
Nonlinear Spring		$u = K(x)$
Nonlinear, Time-Dependent Damper		$u = C(\dot{x}, t)$
Friction Device		$u = a_1 \operatorname{sign}(\dot{x})$
Voigt Element in Series with a Spring		$\dot{u} + a_1 u = a_2 \dot{x} + a_3 x$ $a_1 = \frac{k_1 + k_2}{c}, a_2 = k_2, a_3 = \frac{k_1 k_2}{c}$
Composite Element		$\dot{u} + a_1 u = a_2 \ddot{x} + a_3 x + a_4 x + a_5 \operatorname{sign}(\dot{x}) + a_6$ $a_1 = \frac{k_1}{c_1}, a_2 = c_2 + \frac{dC}{d\dot{x}},$ $a_3 = \frac{dK}{dx} + k_2 - k_1 + \frac{c_2}{c_1} k_1,$ $a_4 = \frac{k_2 k_1}{c_1}, \quad a_5 = \mu \frac{k_1}{c_1},$ $a_6 = K \frac{k_1}{c_1} + C \frac{k_1}{c_1}$

The selection of smooth forms of the isolator force can often be justified on physical grounds. The isolator configurations under consideration may be incapable of responding in a fashion similar to the time-optimal functions. This is particularly true if the optimal isolator is of the on-off or bang-bang type and the candidate isolator configuration is of a simple passive type.

Two aspects of the system identification problem must be considered. One has to do with the measure by which approximate solutions to Eq. (6.8) are to be judged, and the other with how those solutions are obtained.

Measure of the Approximation

There is no reason to expect that the isolator configuration selected is capable of exactly reproducing the time-optimal response functions. That is to say, u_j^*, x_j^*, $\dot{x}_j^*$ generally will not be a particular solution of Eq. (6.8). Let u_j' denote a solution to Eq. (6.8) in which the time-optimal information has been used for some set of values a_{jr}. Then one measure of the approximation is offered by the deviation

$$\Delta_j(t) = |u_j'(t) - u_j^*(t)|. \tag{6.9}$$

The desired a_{jr} can be selected by minimizing a residual function of Δ_j over the time interval of interest for each of the J isolators. We will consider both a least-squares and a maximum deviation form for the residual function. That is, the optimum a_{jr}, say a_{jr}^*, are those for which either

$$H_j = \int_{t_0}^{t_f} [\Delta_j(t)]^2 d(t) \qquad \text{is a minimum} \tag{6.10}$$

or

$$H_j = \max_t |\Delta_j(t)| \qquad \text{is a minimum.} \tag{6.11}$$

The main advantage of expressing the deviation in terms of the isolator forces is that the procedure can then be applied to a multiple-isolator problem, one isolator at a time. The disadvantages have to do with the fact that the u_j^* usually are the least smooth of the time-optimal response functions and, therefore, the most difficult to approximate by continuous functions. Also, minimizing this form of Δ_j may not directly affect the satisfaction of a design constraint imposed on one of the state variables. For example, if a constraint were placed on the relative displacement x_j, it might be more desirable to choose as the deviation

$$\Delta_j(t) = |x_j'(t) - x_j^*(t)|. \tag{6.12}$$

The use of this expression for Δ_j in either Eq. (6.10) or (6.11) would tend to ensure the satisfaction of at least one of the response constraints, in addition to x_j^* being a smoother function. This form of deviation can be used in any SDF system, since then the x_1 and u_1 are directly related. This will also be the case in the general flexible-base situation of Fig. 5.16, since u and x_1 are related through

$$u(t) = -m\ddot{x}_1 - m(\ddot{f} + \ddot{\xi}).$$

The quantity ξ^* can be computed directly from the time-optimal response quantities $\mathbf{x}^*, u^*$.

Other deviations may be formed as the nature of the application suggests. A general, weighted-average form of deviation can be written as

$$\Delta_j = \int_{t_0}^{t_f} \rho(t) U_j^* dt, \tag{6.13}$$

where U_j^* is the value of Eq. (6.8) with both $\mathbf{x}^*$ and u_j^* data used, and $\rho(t)$ is an arbitrary weighting function. Again, Δ_j may be minimized according to either a least-squares or maximum-value criterion.

We will concern ourselves primarily with the deviation defined in Eq. (6.9), since it is most applicable to large, multiple-isolator systems. However, where the form for Δ_j given by Eq. (6.12) can be evaluated conveniently, its use seems preferable on intuitive grounds. Comparative results between the two approaches obtained so far are not conclusive [3].

The procedure is illustrated in Example 10, where we find the optimum design parameters for the SDF linear spring-dashpot isolator under an impulse loading of the base.

System Identification Techniques

We now consider means for obtaining approximate solutions to the general isolator force function, Eq. (6.8). As is evident from Table 6.2, certain types of isolator elements lead to algebraic forms for the force function rather than for a differential equation. In this case, the system identification procedure reduces to a conventional problem of curve matching. Many solution techniques for general, nonlinear forms of Eq. (6.8) have been developed and described in the literature on system identification. In particular, Ref. 3 considers numerical integration, quasi-linearization, so-called method function, and integral equation techniques. For the few examples of simple MDF systems reported, it was found that numerical integration, coupled with a force deviation, Eq. (6.9), and a least-squares residual criterion, Eq. (6.10), offers an acceptable approach.

Consider, for example, an isolator consisting of a parallel linear spring-dashpot element in series with another linear spring. From Table 6.2, the force equation for this type of isolator, Eq. (6.8), is

$$U = \dot{u} + a_1 u - a_2 \dot{x} - a_3 x = 0, \tag{6.14}$$

where the a's are related to the spring rates and the damping coefficient. The j subscript has been dropped for convenience, but the equation is intended to apply to each isolator element of a multiple-isolator system; x and $\dot{x}$ refer to the relative displacement and velocity across the terminal of the isolator, and u and $\dot{u}$ are, respectively, the force and time rate of change of force in the isolator.

The direct evaluation of U requires that $\dot{u}^*$ be available from the time-optimal solution. This will be the case only when u is discretized to ensure its differentiability. While this is possible because of the generality of the formulation of the time-optimal problem, it is of interest here to assume that $\dot{u}^*$ is not available, since higher order and nonlinear forms of Eq. (6.8) for U will necessitate other procedures.

The method of integration is most straightforward. It consists of substituting the time-optimal values $x^*(t)$ and $\dot{x}^*(t)$ into Eq. (6.14) (or, more generally, Eq. (6.8)) to obtain a differential equation that can be integrated directly for u. This is done over a range of admissible a_r, the deviations of Eq. (6.9) are evaluated, and the desired a_r are selected so as to minimize the residual function, either Eq. (6.10) or (6.11). The resemblance of this procedure to the direct synthesis method is only superficial despite the requirement for a nonlinear search code, since only the jth isolator equation, rather than the overall system equations of motion, is being integrated. Also, only explicit constraints on the a_r need be satisfied. Hence, this is essentially a problem of unconstrained minimization.

This procedure was carried out in Ref. 3 for an SDF system with a series-parallel isolator described by Eq. (6.14), and separate results were obtained for deviations based on force, Eq. (6.9), and relative displacement, Eq. (6.12). These are instructive and worth considering in some detail. To complete the setting of the problem, the performance index was based on peak acceleration of the mass, the rattlespace was constrained, and the design parameters a_1, a_2, and a_3 were required to be positive. The base input was the acceleration pulse shown in Fig. 6.3 and the resulting time-optimal response functions $x^*(t)$ and $u^*(t)$ are shown in Fig. 6.4.

Equation (6.14) was solved numerically for u using these data and trial values for the a's. To employ the relative displacement deviation, u and $\dot{u}$ were eliminated in Eq. (6.14) through the relation $u = -m(\ddot{x} + \ddot{f})$, resulting in a third-order differential equation for x. Initial conditions were taken as $x(0) = \dot{x}(0) = u(0) = 0$. For both forms of the deviation, the least-squares residual criterion, Eq. (6.10), was employed using Rosenbrock's Hill Climb algorithm. The following results were obtained for the two solutions.

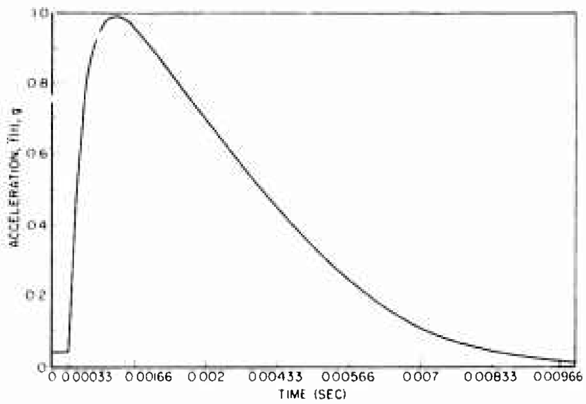


Fig. 6.3. Base motion for system identification example problem.

For $\Delta = |u' - u^*|$:

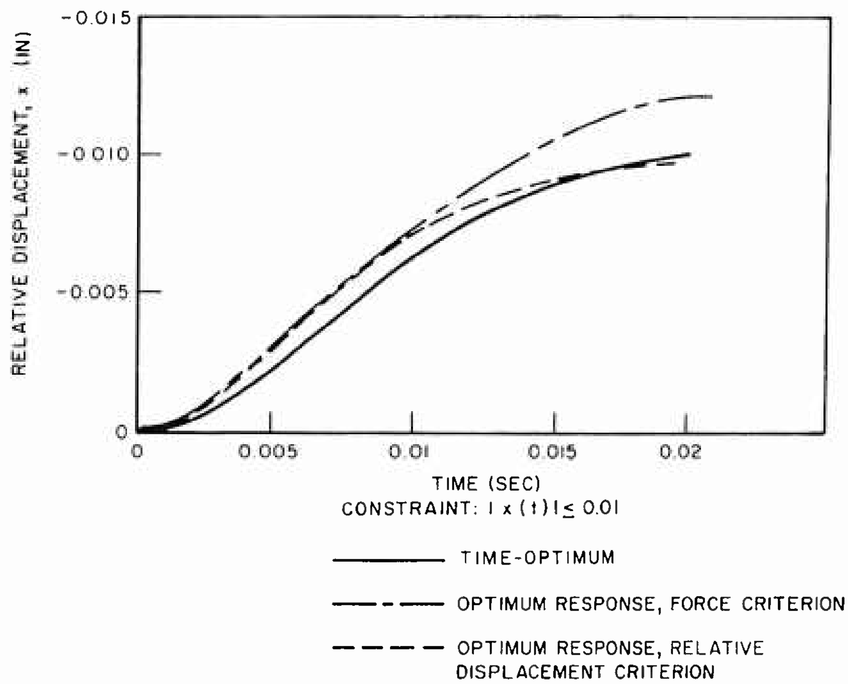
$$a_1^* = 5000\ (\text{sec}^{-1})$$
$$a_2^* = 78{,}000\ (\text{lb/in.})$$
$$a_3^* = 73\ (\text{lb/in.-sec})$$

For $\Delta = |x' - x^*|$:

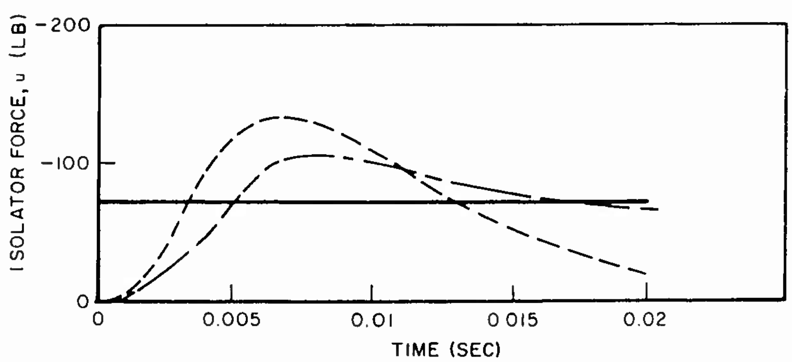
$$a_1^* = 445\ (\text{sec}^{-1})$$
$$a_2^* = 94{,}800\ (\text{lb/in.})$$
$$a_3^* = 142\ (\text{lb/in.-sec})$$

Before discussing the apparently large differences in the design parameters determined according to the two forms of the deviation, we may consider the nature of either solution. Figure 6.5 shows the optimum performance characteristic for this type of isolator as determined by direct synthesis. Shown for comparison purposes is the limiting performance characteristic. The performance of the two isolators corresponding to these two sets of design-parameter values is identified by points in the figure. The isolator design resulting from the force criterion (solid point) exceeds the constraint on rattlespace by about 20 percent. Note, however, that the performance index is essentially optimum for this higher constraint level. The design based on the relative displacement criterion (open point) satisfies the design constraint, and is close to the desired optimum.

The apparent greater success achieved in the latter case probably is a consequence more of the capacity of this type of isolator to approximate $x^*(t)$ than of the measure of the approximation. Because of the reciprocal nature of the time-optimal solution, the same optimum performance characteristics would have resulted had the constraint been imposed on peak acceleration, and rattlespace taken as the performance index. Therefore, since the isolator force is proportional to the peak acceleration, the force form of the deviation also corresponds to a problem constraint. A comparison of the motion and force trajectories for the two isolator



(a) Relative displacement trajectory



(b) Isolator force trajectory

Fig. 6.4. Optimum response trajectories.

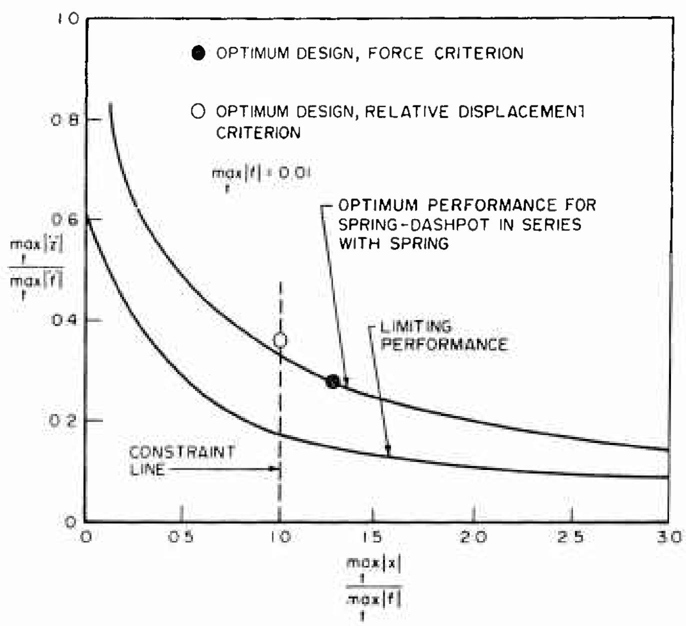


Fig. 6.5. Optimum performance characteristics.

designs with the time-optimal results is shown in Fig. 6.4. It is clear that regardless of the measure of the approximation, this type of isolator cannot match $u^*(t)$ very closely, whereas the displacements differ only slightly.

What is significant is that both approaches yield essentially optimum designs, albeit in the one case for some level of constraint other than that prescribed. This is a crucial point since the indirect synthesis method has value only if it produces a near-optimum design for *some* constraint level. In this event, the performance index will be close to the optimum for the type of isolator being considered† and the desired design can be found by interpolation among the results of several solutions for judicious choices of the constraint levels.

Viewed in this way, the relatively great difference in numerical values between the two optimum designs indicates an insensitivity of the performance index to variations in the parameters. The series spring is indicated to be quite stiff in both solutions. In fact, the response changes hardly at all if this spring is taken to be rigid ($a_2 \to \infty$).

6.2.2 Incompletely Described Environment

The method of indirect synthesis is applicable to a multiple-input description, since a single time-optimal response is determined in conjunction with finding the

†Of course, the optimum performance characteristic for the isolator is unknown in practical situations where the indirect synthesis method will be employed.

limiting performance characteristic. The procedure is exactly as described for a single input except that this would now involve the worst-disturbance input. However, it would have to be verified that the resulting design satisfies the constraints for all inputs.

In contrast, indirect synthesis is not applicable to a class input description, since there is no guarantee that the time-optimal response associated with the worst disturbance of the class will ensure that the constraints are satisfied for other admissible inputs. If, for some reason, however, it were desired to design an isolator for either the worst or best disturbances once these had been found, then the associated time-optimal response could be used in the indirect method as described for a single input.

6.3 Influence of Uncertainty in the Environment

Environments rarely are known with precision, and an isolation system that responds erratically to a disturbance that differs slightly from the one used in the design is of little value. Similarly, the material properties of fabricated components of the system only approximate their mathematical descriptions which are employed in the optimization procedure. Thus, there is concern over the degree of sensitivity of the system response to variations in the design parameters as well.

Here we limit consideration to the effect of variations in the input parameters on the optimum performance characteristics. Three situations are considered: (a) variation of waveform shape, (b) similarly shaped (scaled) waveforms, and (c) extreme members of a prescribed class of inputs. In each instance, unless otherwise indicated, it is assumed that the shock isolation system is fully prescribed.

6.3.1 Variation of Waveforms

The optimum performance characteristics for an SDF, linear spring-dashpot isolator system subject to four markedly different velocity pulses is reported in Ref. 15. The results are shown in the combined plot of Fig. 6.6; the waveforms are shown in Fig. 6.7. The performance index and constraints are normalized to the same characteristic values of the inputs, so that the results are comparable. For each input waveform, the values of k and c are optimized at each constraint level. Thus, Fig. 6.6 indicates the dependence of the optimum performance index on overall waveform characteristics for this type of isolator, rather than the performance sensitivity of a given isolator to variations in the waveform.

Consider Point 1 on Curve II of Fig. 6.6. This indicates that it is possible to design a linear spring-dashpot isolator to attenuate the maximum acceleration of Waveform II by 50 percent when the rattlespace constraint is prescribed at 10 percent of the base displacement. Point 2 indicates that, for Waveform I, this type of isolator can be designed for only a 32-percent reduction of peak base

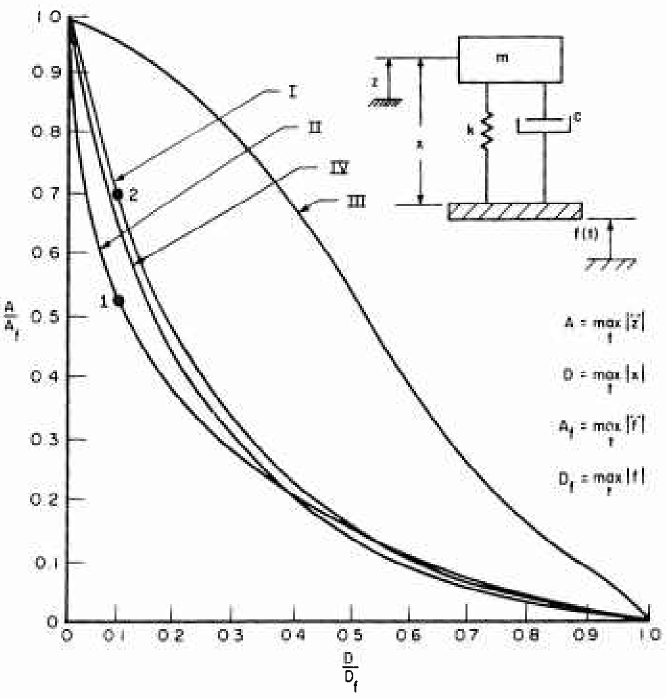


Fig. 6.6. Optimum performance curves for an SDF linear isolator.

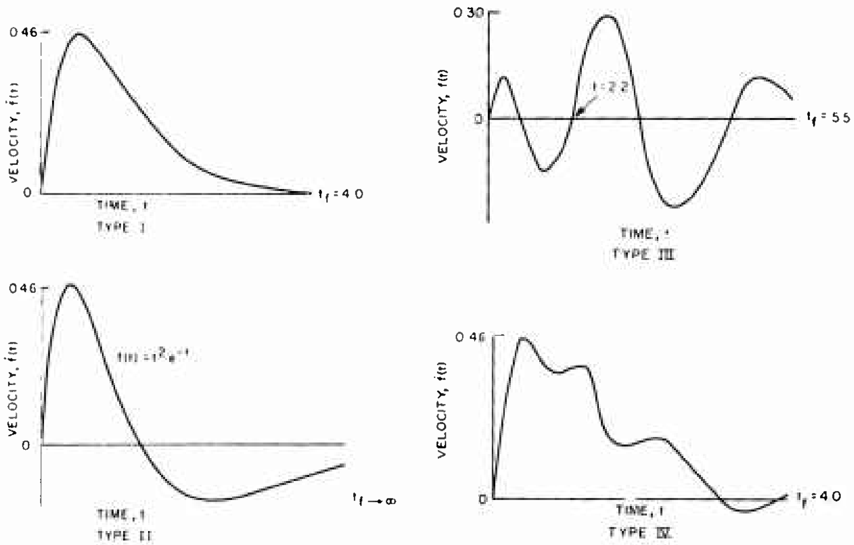


Fig. 6.7. Waveforms of Fig. 6.6.

acceleration for the same constraint level. But this is a different isolator (i.e., different values of k^* and c^*) than that of Point 1. In particular, the performance of the Point 1 isolator to Waveform IV is not indicated in Fig. 6.6 except from the knowledge that it must lie "above" Curve IV. It would, of course, be a simple matter to evaluate the response of a particular optimum isolator design to different waveforms, but that was not done in Ref. 15 and results are not available elsewhere in the literature. While variation-of-parameter studies are straightforward, they are worthwhile only for fairly definite design situations since their generality is limited. The waveforms of Fig. 6.7 are both complicated and arbitrary, and would not justify the extensive computations required to systematically explore their influence on even the simple linear isolator.

We may observe from Fig. 6.6 that the greatest variation in performance results from Waveform III, which possesses the most marked frequency characteristics. In contrast, the limiting performance characteristic for an arbitrary SDF system subject to the same waveforms was presented in Figs. 5.9 through 5.12. A composite plot is shown in Fig. 6.8 which reveals that, even for Waveform III, it is possible to design an isolator (but not a passive linear isolator) to achieve essentially the same performance as for the other waveforms. Some encouragement also may be taken from the close grouping of results for the different waveforms, which suggests that the limiting performance characteristics are rather insensitive to waveform details. This is of significance in regard to the evaluation and improvement potential of shock isolation systems.

6.3.2 Scaling Relations; Small Perturbations of Waveforms

Upper and lower bounds on the optimum performance characteristic, relative to some nominal situation, can be constructed when the input is scaled in a simple manner. The method of construction for an SDF system with rattlespace and peak acceleration criteria is described in Ref. 3. It is assumed that the optimum performance characteristic is known for some nominal input acceleration $\ddot{f}(t)$. Upper and lower bounds to the characteristic curve are sought for the special class of inputs $g(\tau)$ defined by

$$g(\tau) = af(t) \tag{6.15}$$

$$\tau = bt,$$

where a and b are constants.

According to Eq. (6.15), accelerations scale as

$$\ddot{g}(\tau) = ab^{-2}\ddot{f}(t). \tag{6.16}$$

It follows, therefore, that if $D = \max_t |x|$, $A = \max_t |\ddot{z}|$ is a point on the optimum performance characteristic for the input $f(t)$, then aD, $ab^{-2}A$ will be the

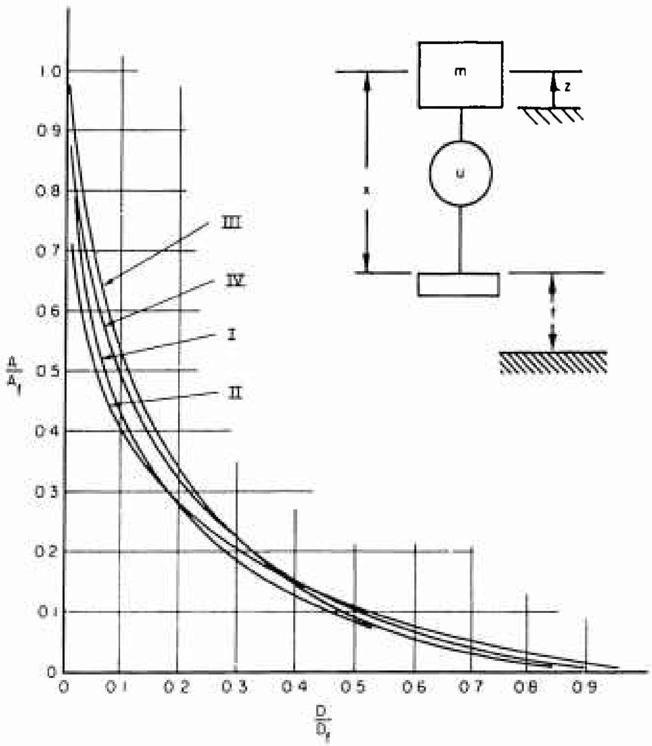


Fig. 6.8. Limiting performance characteristics for an SDF system (waveforms of Fig. 6.7).

corresponding point on the optimum performance characteristic for the scaled input $g(\tau)$. However, the optimum performance characteristic drawn to normalized coordinates, such as those of Fig. 6.8, require no scaling transformation if Eqs. (6.15) are satisfied.

These results have application even where precisely scaled inputs are not expressly involved. For example, the effect of varying the magnitude of the peak input acceleration by a certain amount can be approached by assuming that the modified point lies on a scaled pulse. Figure 6.9 illustrates this approach with regard to determining the influence of a shift in the magnitude and time of occurrence of the maximum input acceleration, i.e., from Point A to Point B. The assumption must be made that Point B is the maximum of the pulse $g(\tau)$ which is related to $f(t)$ by Eq. (6.15). Then, since the values of $\ddot{g}_m$ and τ_m are known (i.e., Point B is specified), the scaling parameters can be computed from

$$b = \frac{\tau_m}{t_m}$$

$$a = \frac{b^2 \ddot{g}_m}{\ddot{f}_m} . \tag{6.17}$$

Therefore, if the rattlespace constraint is D, the minimum peak acceleration caused by the modified pulse $\ddot{g}(t)$ is $ab^{-2}A$, where A is the point on the optimum performance curve for $\ddot{f}(t)$ corresponding to the rattlespace D/a.

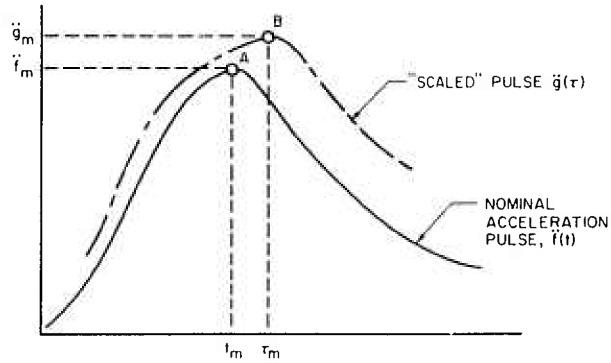


Fig. 6.9. Scaled input waveform.

This approach also can be used to establish upper and lower bounds on the optimum performance characteristic when some feature of the input pulse is uncertain, but bounded. Assume, for example, that the peak acceleration is considered to lie, with equal probability, in a region about its nominal value. If the precise shape of this region and the remaining features of the input are not critical, then each possible value of the peak acceleration can be assumed to lie on a pulse scaled according to Eq. (6.15). This is equivalent to imposing upper and lower bounds on the scaling parameters a, b; i.e.,

$$a^L \leqslant a \leqslant a^U$$

$$b^L \leqslant b \leqslant b^U. \tag{6.18}$$

The nature of the resulting region defining the equally probable values of the (scaled) peak acceleration is suggested in Fig. 6.10. If the upper and lower bounds on peak acceleration and the time of its occurrence are specified, then the bounding values of the scaling parameters are given by

$$\begin{aligned} b^L &= \frac{t_m^L}{t_m} & a^L &= (b^U)^2 \frac{\ddot{f}_m^L}{\ddot{f}_m} \\ b^U &= \frac{t_m^U}{t_m} & a^U &= (b^L)^2 \frac{\ddot{f}_m^U}{\ddot{f}_m}. \end{aligned} \tag{6.19}$$

Then, for each value of the rattlespace D, the minimum peak transmitted acceleration A will lie between the limits

$$a^L(b^U)^{-2}A^L \leqslant A \leqslant a^U(b^L)^{-2}A^U, \tag{6.20}$$

where A^L is the value of A (i.e., for the nominal pulse f) associated with the rattlespace D/a^L, and A^U is the value for A for the rattlespace D/a^U.

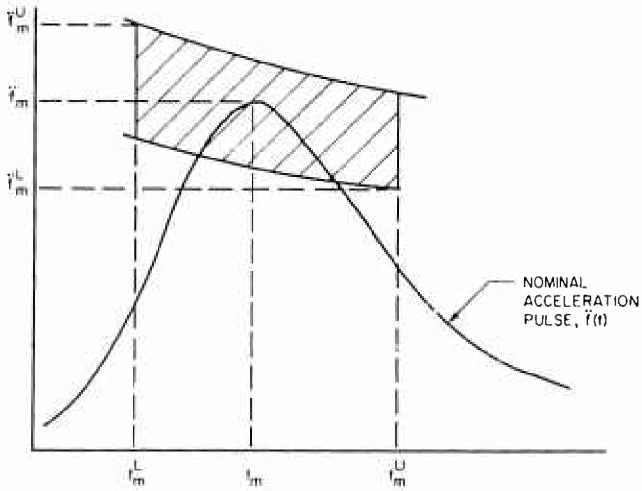


Fig. 6.10. Bounds on scaled input waveform.

An example of this construction is shown in Fig. 6.11, where $a^L = b^L = 1$ and $a^U = b^U = 1.1$. According to Eq. (6.19), this choice of bounds for the scaling parameter corresponds to variations of up to +10 percent in the time and up to +21 percent, −17 percent in the magnitude of the maximum acceleration.

6.3.3 Extreme Disturbance Bounds

One means of expressing the implications of uncertainty implicit in a class description of the input is to establish upper and lower bounds on the performance index, and the corresponding worst and best disturbances. This can be viewed as a special case of the dynamic programming solution for bounding the limiting performance characteristic (discussed in Section 5.2.2). The development is the same except that minimization with respect to the isolator forces is omitted, since these are known functions of the state variables when the system is prescribed.

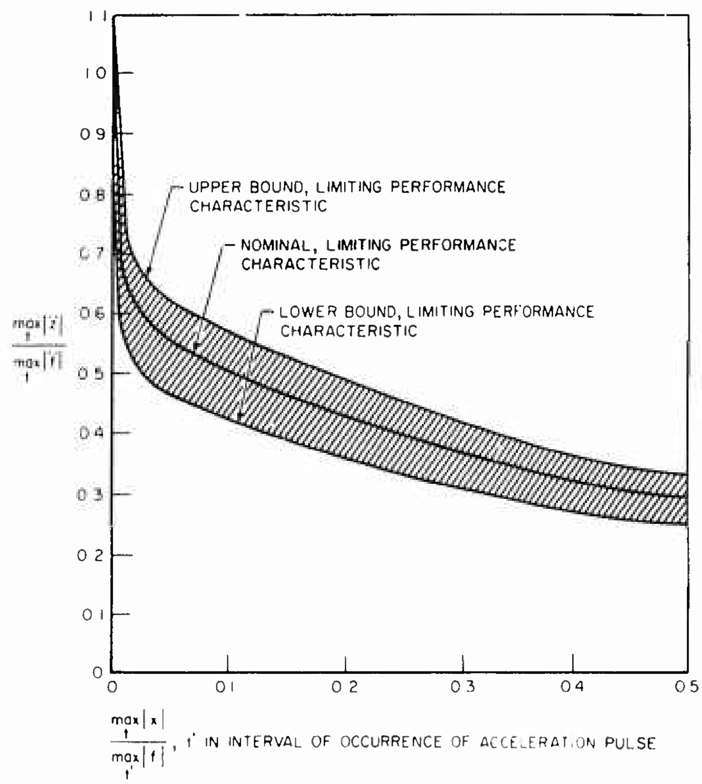


Fig. 6.11. Bounds on optimum performance characteristic for scaled waveforms.

If ψ denotes the performance index, then the desired bounds are given by (see Eq. 5.46)

$$\psi_B^* = \underset{f}{\text{opt}} \; \underset{s}{\max} \; \underset{t}{\max} |h_s(t, \mathbf{x}, f)|, \tag{6.21}$$

where the notation *opt* refers to either a minimization (lower bound) or a maximization (upper bound) with respect to the admissible input f within the prescribed class. The input causing a minimum value is designated the best disturbance and that causing the upper bound, the worst disturbance.

The computational algorithm follows directly from Eq. (5.47) namely,

$$\phi_{I-i+1}(\mathbf{x}_i) = \underset{f_i}{\text{opt}} \max [h(\mathbf{x}_i, f_i), \phi_{I-i}(\mathbf{x}_{i+1})] \tag{6.22}$$

for

$$i = I - 1, I - 2, \ldots, 1,$$

where

$$h(\cdots) = \max_s |h_s(\cdots)|; \qquad s = 1, 2, \ldots, S.$$

The $\mathbf{x}_{i+1}$ are found in terms of the $\mathbf{x}_i$ from the solution of the system equations of motion, Eq. (5.43). The process starts with

$$\phi_1(\mathbf{x}_I) = \max_t |h(\mathbf{x}_I, f_I)|; \qquad t > t_I. \tag{6.23}$$

Upon reaching the Ith stage of the process, the desired bound value is given by

$$\psi_B^* = \phi_I(\mathbf{x}_1), \tag{6.24}$$

where $\mathbf{x}_1$ refers to the prescribed initial state of the system. If the associated best and worst disturbances are desired, another pass forward in time is required for each, as described in Chapter 5.

This solution technique imposes no restrictions on the linearity of the system dynamics or on the form of the performance index. However, as with other dynamic programming formulations, the computational effort rapidly gets out of hand with the increasing size of the system. The determination of extreme disturbance bounds may be formulated as a problem in linear programming provided that the system dynamics of Eq. (5.43), the performance index, and the input class definition all involve the unknown f_i linearly. The best-disturbance solution is exactly the same as described in Section 5.2.2 except that the optimization is with respect to the f_i rather than u_i, and no response constraints are involved.

The worst-disturbance solution is a pure maximization process and, while reducible to LP form, requires a different approach in order to avoid $\psi \to \infty$ as a solution. Let the time at which h_i assumes its maximum value correspond to $i = n$. Then h_n is a known linear function of the f_i for all $i \leqslant n$, which can be maximized as an LP problem. For example, if h is taken as the relative displacement x of the linear spring-dashpot isolator of Fig. 6.1, the results for a piecewise constant approximation to $f(t)$ are

$$x_i = -\sum_{k=1}^{i-1} \ddot{f}_k \frac{\exp[-\beta\omega(t_i - t_k)]}{\lambda\omega} \sin \lambda\omega(t_i - t_k)$$

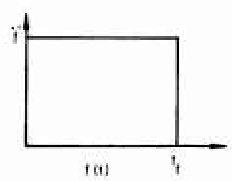
$$\lambda = (1 - \beta^2)^{1/2}, \quad \beta = \frac{c}{c_c}, \quad c_c = 2\omega = 2\left(\frac{k}{m}\right)^{1/2}.$$

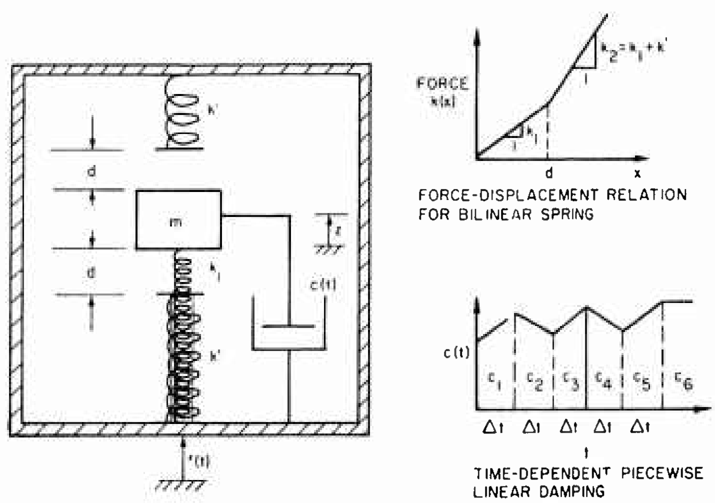
For each $i = n$, the determination of the f_k which maximize x_n is a straight-forward LP problem. If this solution is repeated for a sufficiently large choice of n, the maximum among all the $\max|h_n|$ will be the desired worst-disturbance response, and the associated worst-disturbance input is found in the process. Each maximization is a substantially smaller computation than in the best-disturbance analysis, since the I constraints on ψ are not involved.

Example 8

FORMULATION OF DIRECT SYNTHESIS OPTIMIZATION OF AN SDF ISOLATION SYSTEM

The system consists of a single mass supported within a frame, two concentric helical springs of unequal length (described by parameters k_1, k_2, and d), and a time-dependent viscous damper with piecewise linear force characteristics (described by parameters $c_1, \ldots, c_6$).





There are nine design parameters. The peak transmitted acceleration is chosen as the performance index, and the rattlespace is constrained. In addition, upper and lower bounds are assigned to each of the nine design parameters and to the slope of the damping force-vs-time curve. In all, these amount to 11 constraint relations.

The synthesis problem is to select the nine parameters, k_1, k_2, d, $c_1, \ldots, c_6$, such that

$$\psi = \min_t |\ddot{z}| \text{ is minimized}$$

and the constraint functions

$$C_1^L \leqslant x(t) \leqslant C_1^U$$
$$C_2^L \leqslant k_1 \leqslant C_2^U$$
$$C_3^L \leqslant k_2 \leqslant C_3^U$$
$$C_4^L \leqslant d \leqslant C_4^U$$
$$C_k^L \leqslant c_{k-4} \leqslant C_k^U; \quad k = 5, 6, \ldots, 10$$
$$C_{11}^L \leqslant \frac{c_n - c_{n-1}}{\Delta t} \leqslant C_{11}^U; n = 2, 3, \ldots, 6$$

are satisfied. The state variables x and z are related through the equations of motion

$$m\ddot{z} + c(t)\dot{x} + k(x) = 0$$

and the kinematic condition

$$z = x + f.$$

The mass m, the input motion $f(t)$, the time step Δt, and the bounds C_i^L, C_i^U ($i = 1, 2, \ldots, 11$) are prescribed. The mass is selected to be unity.

Numerical results obtained by the gradient projection method are presented in Ref. 38 for a step pulse input. The following table lists the prescribed values and the optimum design parameters as determined according to three different sets of starting values for the minimization procedure. Whereas the performance index is approximately the same for each of the three solutions, the individual design parameter values vary, particularly the spring rates. This probably indicates a lack of sensitivity of the performance index to the spring rates, but also suggests that only a relative minimum for the performance index may have been found. In any event, any of these three designs represents an improvement in performance of about 30 percent relative to a constant-rate spring and a constant-rate damper design.

Constraints and Optimum Design Parameters

Parameter†	Lower Bound, C_i^L	Upper Bound, C_i^U	Optimum Parameter Values		
			Solution 1	Solution 2	Solution 3
$x(t)$	1.1	1.1			
k_1	4.5	700	368	31	42
k_2	k_1	700	368	670	202
d	0	1.1	0.0	0.5	0.5
c_1	0	100	81.7	84.9	88.2
c_2	0	100	81.7	65.0	68.2
c_3	0	100	41.7	45.3	48.2
c_4	0	100	25.4	33.3	34.2
c_5	0	100	14.6	21.5	25.6
c_6	0	100	19.8	22.1	35.7
$\frac{c_n - c_{n-1}}{\Delta t}$	-1600	1600			
min ψ			635	626	640

† $\ddot{f}$ = -1000; t_f = 0.05; Δt = 0.125; all units consistent.

Example 9

DIRECT SYNTHESIS OF AN SDF ISOLATOR FOR AN INPUT CLASS DESCRIPTION

This example is taken from Ref. 3, which considers an SDF linear spring-dashpot isolator subject to the bounded class of base-input acceleration pulses shown in the figure on the next page.

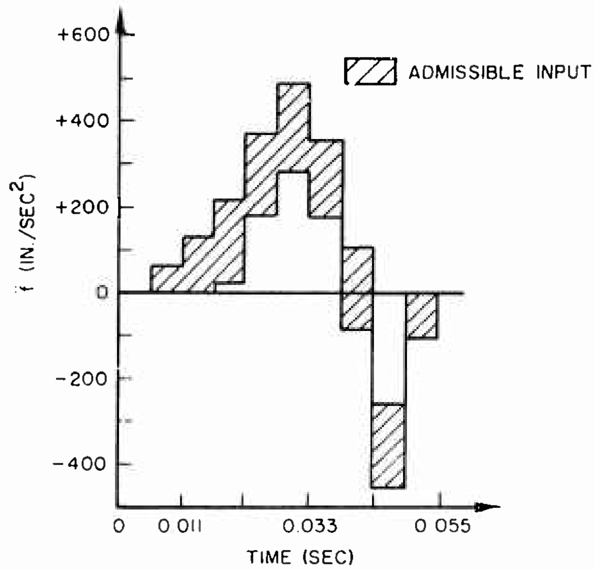
The bounds to the input class are represented by crude piecewise constant approximations for the purposes of this illustration. The performance index is taken to be the peak acceleration of the mass, and the rattlespace is constrained. In addition, both the spring rate k and the damping coefficient c are restricted to a prescribed range of positive values. More specifically, the optimum design problem is to select values of k and c from among the following range:

$$13{,}000 \text{ lb/in.} \leqslant k \leqslant 15{,}000 \text{ lb/in.}$$

$$40 \text{ lb-sec/in.} \leqslant c \leqslant 60 \text{ lb-sec/in.},$$

so that

$$\max|x| \leqslant 0.4 \text{ in.}$$



and

$$\psi = \max_t |\ddot{z}| \text{ is minimized}$$

for any admissible $\ddot{f}(t)$ among the prescribed class. The isolated mass is taken to be unity.

The solution is started by selecting an admissible trial set of (k, c), say the lower bound values. Whether this set of parameters is actually acceptable depends on the rattlespace constraint not being violated for any $\ddot{f}(t)$. This requires a worst-disturbance analysis to find the largest rattlespace possible for the loading class; that is, we must find

$$D = \max_{\ddot{f}} \max_t |x|.$$

Since the system is linear, the LP solution to this worst-disturbance analysis described in Section 6.3.3 can be used. If we find that $D \leqslant 0.4$, then the choice of (k, c) is acceptable since the rattlespace constraint will be satisfied whatever input within the class is experienced by the system. However, if $D > 0.4$, then a new choice of (k, c) must be made until this constraint is satisfied.

Once an acceptable set (k, c) is determined, the value of the performance index

$$\psi_B = \max_{\ddot{f}} \max_t |\ddot{z}|$$

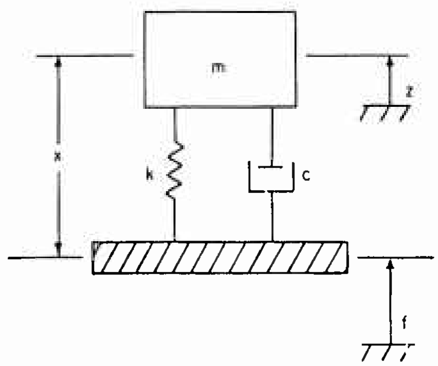
is evaluated. Observe that this requires another worst-disturbance analysis and, in general, will lead to a different member of $\ddot{f}(t)$ than that found for the largest rattlespace. Since the response function $\ddot{z}$ again is linear, the LP solution can be used.

For each acceptable set (k, c), the associated ψ_B constitutes a point on the response surface. A mathematical search procedure can then be used to determine that set (k^*, c^*) for which ψ_B is a minimum. Rosenbrock's Hill Climb Method [40] was used in Ref. 3. It was found that $\psi_B^* = 581$ in./sec^2, corresponding to $k^* = 13{,}260$ lb/in. and $c^* = 59.4$ lb-sec/in.

Example 10

INDIRECT OPTIMUM SYNTHESIS OF AN SDF ISOLATOR SYSTEM

We consider the SDF linear spring-dashpot isolator whose base undergoes an impulsive loading characterized by the initial velocity.



The performance index is selected to be the peak acceleration of the mass and the rattlespace is constrained. The spring rate k and damping coefficient c are both required to be positive, and the system is to be overdamped, i.e., $c \geqslant 2\sqrt{km}$. This problem was solved analytically by the method of direct synthesis in Section 6.1.1, where it was found that

$$\psi^* = \frac{2V^2}{eD}$$

$$k^* = \frac{V^2 m}{e^2 D^2}$$

$$c^* = 2\sqrt{k^* m} = \frac{2Vm}{eD} ,$$

in which D is the rattlespace constraint and $e = 2.718$.

The indirect synthesis method requires the solution to the time-optimal response quantities $u^*(t)$, $x^*(t)$, and $\dot{x}^*(t)$. For the impulse loading case, these quantities are most easily found by the graphical method described in Section 5.1.1. The results are

$$u^*(t) = -Am$$

$$x^*(t) = \frac{1}{2}At^2 - Vt$$

$$\dot{x}^*(t) = At - V,$$

where

$$A = \frac{V^2}{2D}.$$

These functions are applicable only during the time interval $0 \leqslant t \leqslant V/A$. A unique solution cannot be found beyond this interval as the problem is stated, but this need not concern us here.

The isolator force function is

$$u_1 = u = kx + c\dot{x},$$

so that the approximation is

$$u' = kx^* + c\dot{x}^* = k\left(\frac{1}{2}At^2 - Vt\right) + c(At - V).$$

The deviation is formed from Eq. (6.9) and the appropriate residual function minimized to determine k and c. This can be carried out analytically, although, in general, a numerical procedure utilizing a discrete formulation for the residual function would have to be employed.

The values of k^* and c^* as determined in this manner are compared with the exact values in the table below for several forms of the residual function. The integral equation method [3] provides the best results but is not applicable to an isolator-by-isolator design approach for large systems; hence, it will not be discussed. The least-squares approximation of Eq. (6.10) gives quite satisfactory results.

Comparison of Direct and Indirect Synthesis for an Overdamped Linear Spring-Dashpot Isolator

Residual Function†	Percentage Error in	
	k^*	c^*
Least Squares, Eq. (6.10)	- 5.0	- 2.5
Min-Max, Eq. (6.11)	+26.6	+12.5
Average‡	+15.6	+ 7.5
Integral Equation Method [3]	+ 0.07	+ 0.04

†The first three entries are based on Eq. (6.9) for Δ.
‡Δ^2 replaced by $|\Delta|$ in Eq. (6.10).

Chapter 7

HARMONIC VIBRATION ISOLATION SYSTEMS

Vibration isolation refers to the mitigation of disturbances that are oscillatory in nature and extend over relatively long periods. Conventionally, vibration isolation is thought of as the attenuation of a steady-state motion. While the excitation is prescribed as a function of time (for deterministic representation), the equations of (steady-state) motion are not of the initial value type as for shock isolation. Hence, synthesis in the time domain, as was used to determine the limiting performance characteristic for shock isolators, is not directly applicable. Moreover, the vibration isolation designer usually cannot settle for a motion possessing a single frequency, but must investigate system performance over a range of frequencies. Thus, within the context of this monograph, the optimum design problem for vibration isolation generally belongs to the class of unprescribed inputs, for which the synthesis approach is by the direct method (Section 6.1).

Most of the literature deals with systems of relative simplicity for which closed-form solutions for the steady-state motions can be obtained. This reduces the effort associated with the direct synthesis method and is the basis for the well-known examples of the tuned and optimally damped vibration absorber. The literature dealing with a more computationally oriented approach to larger and more complex systems is meager.

In this chapter we limit our consideration to harmonically excited systems and present some recent work on (a) performance bounds for discrete frequency excitations, (b) direct synthesis of a damped linear isolator, and (c) computationally oriented synthesis of complex MDF systems with inputs possessing a range of possible frequencies and amplitudes. Performance criteria are based on peak response variables such as acceleration and displacement.

7.1 Limiting Performance Characteristics

Reference 41 establishes with the calculus of variations the limiting performance characteristic of an SDF isolator for harmonic excitation according to peak acceleration and rattlespace criteria. We will now derive these same results from an argument based on kinematics.

The base displacement is denoted by $f(t)$, the absolute displacement of the isolated mass by $z(t)$, and the relative displacement of the isolator $x(t)$ is defined by

$$x(t) = z(t) - f(t). \tag{7.1}$$

We choose $f(t)$ to be

$$f(t) = f_m \sin \Omega t. \tag{7.2}$$

For steady-state motion, the mass will respond with the frequency Ω so that

$$z(t) = x_m \sin \Omega t + f_m \sin \Omega t, \tag{7.3}$$

where x_m is the maximum relative displacement of the mass, or the rattlespace. A relationship between rattlespace and peak acceleration of the mass $\ddot{z}_m$ is found from Eq. (7.3) to be

$$-\ddot{z}_m \sin \Omega t = x_m \Omega^2 \sin \Omega t + f_m \Omega^2 \sin \Omega t.$$

Since $f_m \Omega^2$ is the maximum base acceleration, we may denote this by $\ddot{f}_m$, and the inequality

$$\left| \frac{\ddot{z}_m}{\ddot{f}_m} \right| + \left| \frac{x_m}{f_m} \right| \geqslant 1 \tag{7.4}$$

holds. Equation (7.4) implies that admissible values of rattlespace and peak acceleration, for any frequency of excitation, may lie anywhere in the first quadrant of the $(x_m/f_m, \ddot{x}_m/\ddot{f}_m)$ plane with the exception of the triangular region bounded by the coordinate axes and the line joining the points (0, 1) and (1, 0). This is shown in Fig. 7.1, where the dashed lines pertain to the linear isolator we will discuss.

7.2 Optimum Synthesis of a Damped Linear Isolator

We consider the linear SDF system shown in Fig. 7.2 and seek the values of spring rate k and damping coefficient c that minimize rattlespace subject to a constraint on the peak transmitted acceleration. The base input is assumed to be harmonic with frequency Ω. The equations of motion are

$$m\ddot{z} + c\dot{x} + kx = 0$$

$$z = x + f_m \sin \Omega t. \tag{7.5}$$

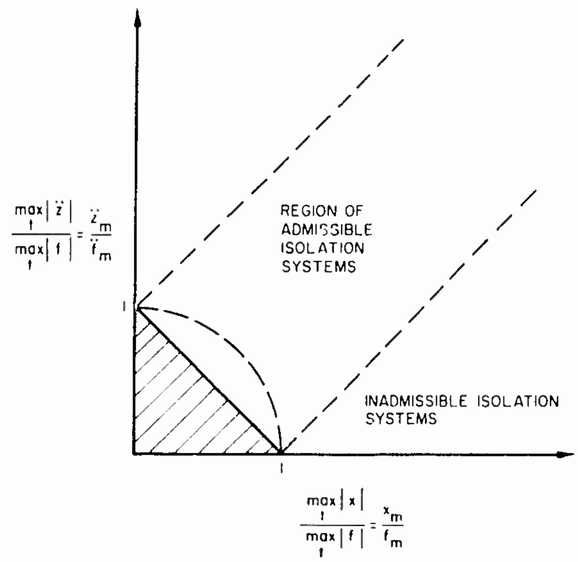


Fig. 7.1. Limiting and optimum linear performance characteristics for an SDF system.

The steady-state solution is

$$x(t) = f_m \eta [(1-\eta)^2 + 4\xi\eta]^{-1/2} \sin(\Omega t - \phi), \tag{7.6}$$

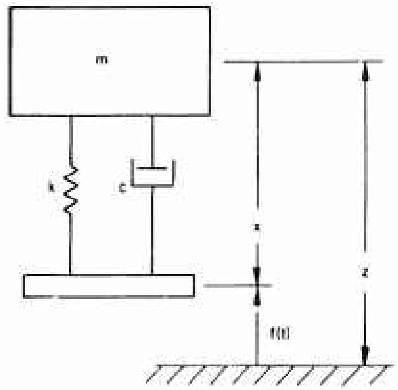
where

$$\left.\begin{aligned} \eta &= \left[\frac{\Omega}{\omega}\right]^2; \quad \omega = \sqrt{\frac{k}{m}} \\ \xi &= \frac{c^2}{4km} \\ \tan\phi &= \frac{2\xi\eta}{1-\eta}. \end{aligned}\right\} \tag{7.7}$$

Normalized forms for the rattlespace x_m and peak acceleration $\ddot{z}_m$ are given by

$$\begin{aligned} \frac{x_m}{f_m} &= \eta[(1-\eta)^2 + 4\xi\eta]^{-1/2} \\ \frac{\ddot{z}_m}{\ddot{f}_m} &= (1+4\xi\eta)^{1/2}[(1-\eta)^2 + 4\xi\eta]^{-1/2}, \end{aligned} \tag{7.8}$$

where $\ddot{f}_m = f_m\Omega^2$.



GIVEN. PRESCRIBED BASE MOTION, f(t)
PERFORMANCE INDEX: $\psi = \max_t |x|$
CONSTRAINT: $\max_t |\ddot{z}| \leq A$, A PRESCRIBED
FIND k AND c TO MINIMIZE ψ

Fig. 7.2. Linear spring-dashpot isolator system.

Equations (7.8) provide a point on the optimum performance characteristic for each set (η, ξ), or equivalently for k and c, associated with a prescribed excitation frequency Ω. The spring constant k, damping ratio c, and ratio of critical damping for any combination of criteria are

$$\frac{1}{\eta} = \frac{k}{m\Omega^2} = \frac{X^2 + Z^2 - 1}{2X^2}$$

$$\lambda = \frac{c^2}{m^2\Omega^2} = \frac{(X + Z + 1)(X + Z - 1)(X - Z + 1)(Z - X + 1)}{4X^4} \tag{7.9}$$

and

$$\xi = \frac{c^2}{4km} = \frac{\lambda\eta}{4},$$

where

$$X = \frac{x_m}{f_m}; \qquad Z = \frac{\ddot{z}_m}{\ddot{f}_m}.$$

All positive values of k and c lie in a semi-infinite region of the (X, Z) plane within the first quadrant, bounded by a quarter circle centered at the origin and the two lines emanating from the points (0, 1) and (1, 0) inclined 45° to the

coordinate axes. This region is shown by the dashed lines in Fig. 7.1. Values of X, Z are plotted in Figs. 7.3, 7.4, and 7.5 [15] for constant values of η, λ, and ξ, respectively. The straight-line boundaries correspond to the limiting case $c \to 0$, $k > 0$; the points on the quarter circle are for the limiting case $k \to 0$, $c > 0$.

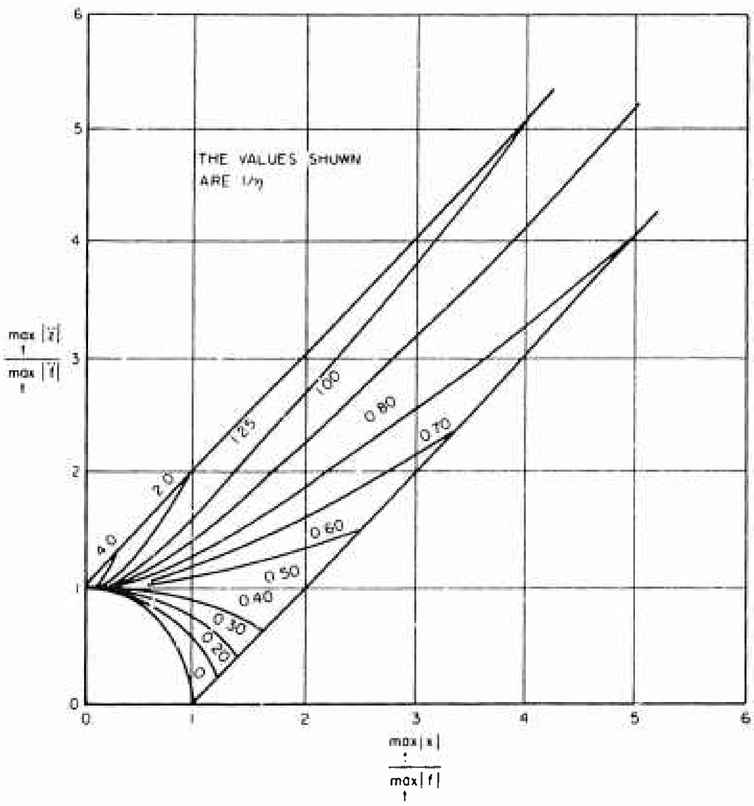


Fig. 7.3. Optimum performance characteristics for constant $1/\eta$.

7.3 Incompletely Described Environment

Harmonic environments are often incompletely prescribed in that the frequencies or amplitudes may vary over a range of values. Sometimes harmonic disturbances are characterized by a frequency-vs-amplitude spectrum of the sort shown in Fig. 3.2, where, for each amplitude level, there is a range of possible frequencies and vice versa. From an optimization point of view, the literature deals mostly with the problem of optimum damping, wherein the quantity and distribution of damping are sought such that the peak response of the system is minimized over a range of input frequencies. The input amplitude is assumed

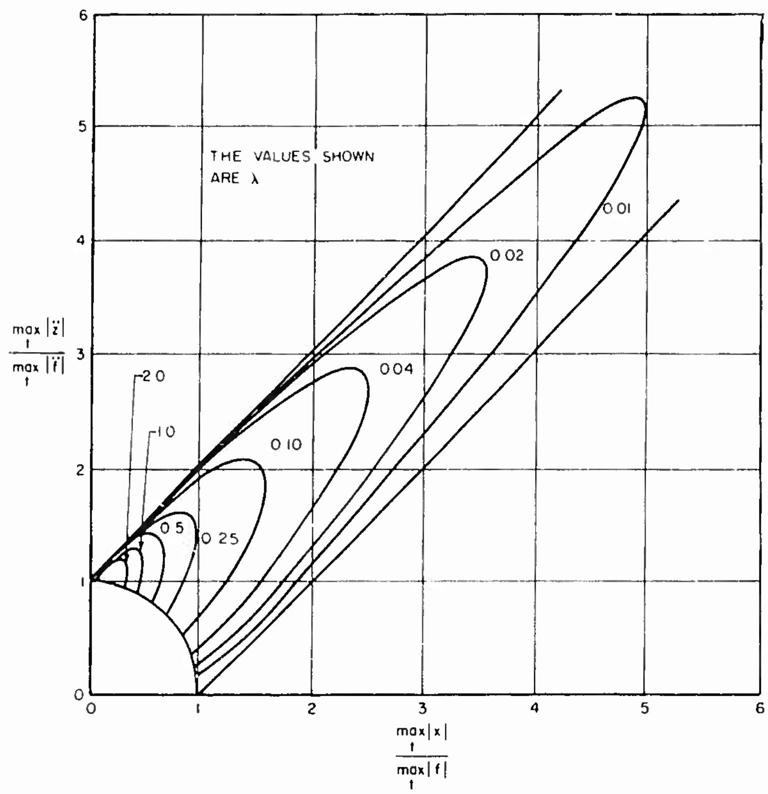


Fig. 7.4. Optimum performance characteristics for constant λ.

constant. In 1928, Ormondroyd and Den Hartog [42] introduced the concept of optimum damping in connection with the study of a linear two-degree-of-freedom system with viscous damping. They found that, for a harmonic input of variable frequency, there are two frequencies at which the response is independent of the damping coefficient. On a plot of maximum displacement as a function of frequency, these two frequencies are termed the *fixed points.* They define optimum damping as *that value of the damping coefficient for which the response curve passes through the higher of the two fixed points with zero slope.* The system is said to be optimally damped over that frequency range for which the maximum displacement does not exceed the highest fixed-point value.

Since this early work, many interesting variations of this concept have been investigated [43-74]. Generally these have dealt with simple systems possessing several dampers and MDF systems with single dampers. Some of these efforts are summarized in the annotated bibliography.

An MDF, multiple-parameter, optimum damping problem is formulated in Ref. 75 for computational solution. The system is linear, stable, and strictly

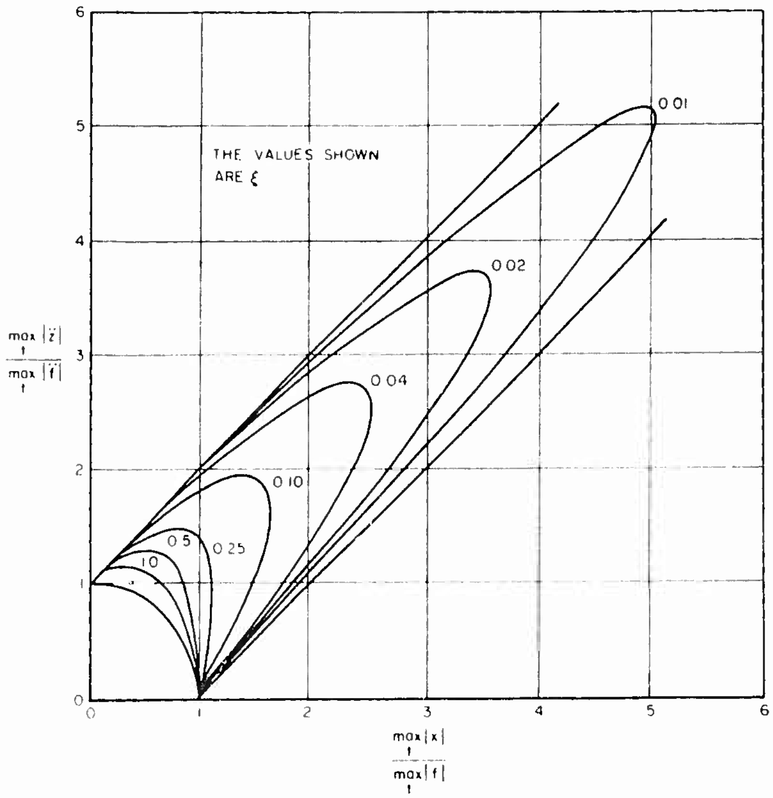


Fig. 7.5. Optimum performance characteristic for constant ξ^2.

dissipative, and the loading is harmonic with amplitude fixed and frequency variable; optimum damping is defined as that set of damping rates that minimizes the maximum displacement of some point in the system. As in the case of the simple systems considered previously, the time variable is eliminated from the problem at the outset and the system equations are nondifferential relations containing the input frequency Ω and the damping parameters. The optimum damping problem becomes a min-max problem in that an expression for displacement is to be maximized with respect to frequency and minimized with respect to the damping rates. The solution is obtained by performing a single variable maximization over the range of admissible frequencies at each iteration of a computational minimization scheme designed to select the damping parameters. The procedure is reasonably straightforward, since no response constraints are imposed.

Numerical results are presented in Refs. 75 and 76 for a five-degree-of-freedom model of a vehicle subject to a sinusoidal disturbance representing a rough roadway. The dynamic system is shown schematically in Fig. 7.6. The main vehicle structure is represented by the rigid mass m_2 which is permitted

two degrees of freedom, the vertical displacement x_2, and angular rotation x_3. The suspension system is modeled by a linear spring-dashpot arrangement (k_2, c_2 and k_3, c_3) connected at either end of the vehicle frame to the wheel-axle masses m_3 and m_4. Tire flexibility is included as the linear springs k_4 and k_5. The driver is modeled by mass m_1 and the spring-dashpot (k_1, c_1). A numerical search procedure is employed to determine the damping factors c_1, c_2, and c_3 that minimize the driver displacement x_1 (expressed as a ratio of the roadway amplitude x_0) over a specified frequency interval $\Delta\Omega$.

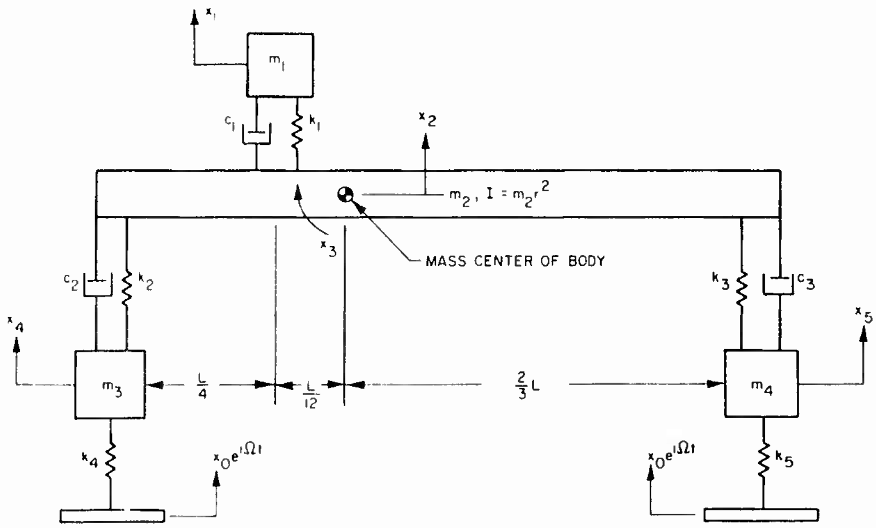


Fig. 7.6. Lumped-parameter model of a vehicle traveling over a sinusoidal road.

The optimum damping problem can be generalized to encompass a large class of harmonic vibration isolation systems which contain design parameters other than just damping rates. Assume that the system is subject to a set of inputs, each member of which is of the form $f(t) = f_m \exp(i\Omega t)$ where amplitude f_m and frequency Ω can, with equal probability, be any point in a prescribed region, as in Fig. 3.2. Suppose, further, that the design is to be based on the worst occurrence to the system. Then the optimum design-parameter synthesis problem is to select the design parameters such that the maximum of a response function is minimized while imposed constraints are not violated. This requires a worst-disturbance analysis of the system, similar to that discussed in Chapter 6 for shock isolation systems, although here the uncertainties are reflected in the (f_m, Ω) class definition rather than in terms of the input waveform details.

Suppose the isolator configurations are prescribed as functions of the desired design parameters; i.e., $u_j(x, \dot{x}, a_{jr})$, where $j = 1, 2, \ldots, J$ and $r = 1, 2, \ldots, R_j$ are known. The performance index for the worst-disturbance design can be written

$$\psi = \max_{f_m, \Omega} \max_s |h_s(u_j)|; \qquad s = 1, 2, \ldots, S, \tag{7.10}$$

where the overall maximization is with respect to admissible combinations of f_m and Ω. The problem statement now is to determine the design parameters a_{jr} ($r = 1, 2, \ldots, R_j$) such that the performance index is minimized and the constraints satisfied for all potential disturbances. This minimum performance index ψ^* is written

$$\psi^* = \min_{a_{jr}} \max_{f_m, \Omega} \max_s |h_s(u_j)|; \qquad s = 1, 2, \ldots, S. \tag{7.11}$$

This is the standard min-max problem. From a mathematical programming viewpoint it can be approached by means of a worst-disturbance analysis at each iteration of a minimization scheme. This analysis must be applied both to the constraint functions and the response variables that make up the performance index. Thus, at each stage of the minimization procedure, i.e., for each trial set of a_{jr},

$$\max_{f_m, \Omega} C_k$$

must be computed and compared to the prescribed bounds to ensure that the candidate set of a_{jr} does not lead to a violation of the constraints for any admissible input. The value

$$\max_{f_m, \Omega} \max_s |h_s(u_j)|$$

is also calculated and used as the current value of ψ. The logic of the minimization technique is used to select the next trial set of a_{jr} and the procedure repeated until $\min_{a_{jr}} \psi$ is achieved. The admissibie values of f_m, Ω can be considered as constraints in the worst-disturbance analysis, whereas the parameters a_{jr} are usually bounded.

In general, the system equations which enter the problem through the response variables of h_s and C_k are differential equations. However, in many important cases, such as linear spring-mass-dashpot systems, the system equations reduce to algebraic or transcendental relations independent of time. The analysis is considerably simplified, since differential equations need not be repetitively solved. Indeed, powerful synthesis techniques developed for static structural systems can

be brought to bear on this problem. The required worst-disturbance analysis may take the form of a nonlinear programming problem in which the maximum of a function, i.e., h_s or C_k, is to be found subject to constraints on f_m and Ω which define the class of disturbances. In general, this would mean that a nonlinear maximization programming problem is to be solved at each iteration of a nonlinear minimization programming problem. Clearly, this can become a formidable task for large systems. The literature contains no results for such optimization problems, but the approach is clear.

Chapter 8

RANDOM VIBRATION ISOLATION SYSTEMS

Random disturbances appear as complicated time-varying functions that may exhibit wide, irregular variations in amplitude and frequency. Both the input disturbances and the system response must be given statistical characterizations and, as we would expect, this complicates the optimum design problem. No encompassing methodologies are available for optimizing realistic isolation systems under general random environments, although related literature from control theory on the optimization of stochastic processes is becoming quite extensive. While this undoubtedly will form the basis for advancements in isolation system optimization, we do not consider it appropriate for inclusion in the monograph. Consequently, this chapter is of limited scope.

Some studies directly applicable to isolation system design have dealt in a preliminary way with performance indices based on maximum values of the response variables [6, 22], but detailed solutions are available only for expected mean-square values and related quadratic optimization criteria associated with linear systems dynamics [77]. We will restrict ourselves to such systems and to input disturbances that are stationary random functions of time as characterized by the power spectral density (Chapter 3). However, the solution techniques generally are applicable to other stationary disturbances.

The organization of this chapter is similar to that of the previous one in that we deal first with the limiting performance characteristic, then proceed to design-parameter synthesis for given isolator configurations.

8.1 Limiting Performance Characteristic

Limiting performance characteristics of SDF systems are reported in the literature for several inputs and are based on either expected mean-square values or the probability of exceeding selected response levels. These indices are, respectively (see Chapter 2),

$$\psi = \mathrm{E}[\ddot{z}^2] + \rho \mathrm{E}[x^2] \tag{8.1}$$

and

$$\psi = x_0^2 + \rho \ddot{z}_0^2. \tag{8.2}$$

Here, $\ddot{z}$ is the acceleration of the mass; x the relative displacement of the isolator (Fig. 8.1); E[ ] denotes expected value; and x_0 and $\ddot{z}_0$ are values of x and $\ddot{z}$ for which the probabilities of $|x| \leqslant x_0$ and $|\ddot{z}| \leqslant \ddot{z}_0$ are both equal to a prescribed value, $1 - P$, over some time interval of interest, T. The quantity ρ is a weighting tor which emphasizes either the relative displacement or acceleration according to whether its value is large or small.

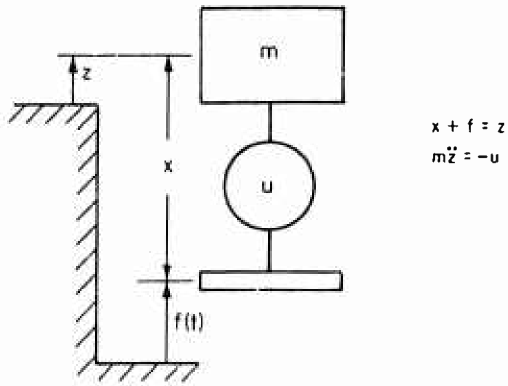


Fig. 8.1 SDF isolation system.

A lower bound on the performance index can be found by the Wiener filter method [78], provided that the random input can be characterized by its power spectral density and no constraints are imposed on the response variables. A relationship between the elements of the performance index can be found which defines the limiting performance characteristic for the SDF system. While the method (a) is restricted to linear systems, (b) is limited as to input forms, and (c) does not appear promising for larger systems, it is still the most advanced solution technique available. Thus, we will only sketch the solution method and the more significant results; the reader is referred to the several references for details. Specific limitations of the applicability of this approach to isolation systems are presented in Ref. 24.

The Wiener filter method makes use of Laplace transform techniques and the notion of an optimum transfer function. We will denote the transformed response quantities and inputs by capital letters and the transform time variable by s. Then the expected mean-square value of a response function Y is

$$\mathrm{E}[y^2] = \frac{1}{i}\int_{-i\infty}^{i\infty} Y_f(s)Y_f(-s)S_f(s)ds, \tag{8.3}$$

where $Y_f(s)$ is the transfer function defined by

$$Y(s) = Y_f(s)F(s) \tag{8.4}$$

and $S_f(s)$ is the power spectral density of f. Reference 78 contains extensive tables for the evaluation of Eq. (8.3).

Reference 1 presents results for the performance index of Eq. (8.1) and a random input whose power spectral density is

$$S_f(t) = \frac{A}{\Omega^2}, \tag{8.5}$$

where A is a prescribed constant. This is termed a white noise disturbance. Since $s = i\Omega$,

$$S_f(s) = \frac{-A}{s^2}.$$

The transfer functions relating the relative displacement and acceleration of the mass are

$$X(s) = [W(s) - 1]\frac{A}{s^2}$$

and (8.6)

$$\ddot{Z}(s) = -W(s)A.$$

The optimum value of the function $W(s)$ is found in Ref. 1 to be

$$W^*(s) = \frac{\beta^2}{s^2 + \sqrt{2}\,\beta s + \beta^2}, \tag{8.7}$$

where $\beta = \rho^{1/4}$.

Equation (8.3) evaluated for $W^*(s)$ results in

$$E[x^2] = \frac{3\pi}{\sqrt{2}}\frac{A}{\beta}$$

and (8.8)

$$E[\ddot{z}^2] = \frac{\pi\beta^3 A}{\sqrt{2}}.$$

Eliminating β yields the desired expression for the limiting performance characteristic

$$(\mathrm{E}[x^2])^3\mathrm{E}[\ddot{z}^2] = \frac{27\pi^4}{4} A. \tag{8.9}$$

This relationship will appear as a family of parallel straight lines on a log-log plot.

Similar results are derived in Ref. 7 for the performance index of Eq. (8.2). A version of the Wiener filter method is used to minimize ψ in which x_0^2 and $\ddot{z}_0^2$ are expressed in terms of expected values. It is shown for small P, large T, and disturbances with gaussian distributions and zero mean values, that y_0^2 of a response function y is given by

$$y_0^2 = 2\mathrm{E}[y^2]\left\{\log\left[\frac{T}{\pi P T_0}\right]\right\} + \log\left\{\mathrm{E}[\dot{y}^2]\,\frac{T_0}{\mathrm{E}[y^2]}\right\}, \tag{8.10}$$

where T_0 is an arbitrary time unit. Limiting performance characteristics are computed in Ref. 7 for selected values of P, ρ, T/T_0 and are compared with those based on the former performance index. The forms of the optimum transfer functions are compared in Table 8.1 for several input spectral densities.

Table 8.1. Optimum Transfer Functions

Input Spectral Density, $S_f(s)$	Optimum Transfer Function $W^*(s)$†	
	$\psi = \mathrm{E}[x^2] + \rho\mathrm{E}[\ddot{z}^2]$	$\psi = x_0^2 + \rho\ddot{z}_0^2$
$-\dfrac{A}{s^2}$	$\dfrac{\beta^2}{s^2 + \sqrt{2}\beta s + \beta^2}$	$\dfrac{a_1}{s^3 + a_2 s^2 + a_3 s + a_1}$
$\dfrac{A}{s^4 - \alpha_1 s^2 + \alpha_2}$	$\dfrac{a_1 s + a_2}{s^2 + \sqrt{2}\beta s + \beta^2}$	$\dfrac{a_1 s + a_2}{s^3 + a_3 s^2 + a_4 s + a_5}$
$\dfrac{A}{s^4}$	$\dfrac{2\beta s + \beta^2}{s^2 + \sqrt{2}\beta s + \beta^2}$	$\dfrac{a_1 s + a_2}{s^3 + a_3 s^2 + a_1 s + a_2}$
$-\dfrac{A}{s^6}$	$\dfrac{-s^2 + \sqrt{2}\beta s + \beta^2}{s^2 + \sqrt{2}\beta s + \beta^2}$	$\dfrac{a_1 s^2 + a_2 s + a_3}{s^3 + a_1 s^2 + a_2 s + a_3}$

†α_i, a_i, and β are known constants.

Systems optimized according to criteria similar to Eqs. (8.1) and (8.2) do not necessarily respond optimally on the basis of other criteria. Interesting examinations of this problem are found in Ref. 1 and 79.

The limiting performance characteristic can be improved upon if the system is permitted to sense the input before it is actually encountered. Reference 80 treats this concept of preview sensing in the context of a vehicle traversing a roadway of spectral density

$$S_f = \frac{AV}{\Omega^2}, \tag{8.11}$$

where V is the constant vehicle velocity and A is a property of the roadway. The limiting performance characteristic as a function of the preview time $T_1 = L/V$ is shown in Fig. 8.2; the preview distance L is defined in the insert of the figure. A substantial improvement in isolation is seen to result from inclusion of a preview sensor.

The Wiener optimization procedure is applied in Ref. 10 to the two-degree-of-freedom, single-isolator system shown in Fig. 8.3. This is equivalent to the flexible-base problem considered in Example 5. The input spectral density is the same as Eq. (8.11) and the performance index is of the weighted expected mean-square value type shown in Eq. (8.1). Typical results for the limiting performance characteristic (without preview) are shown in Fig. 8.4 for the mass ratio $m_1/m_2 = 0.1$.

8.2 Optimum Design-Parameter Synthesis

Optimum design-parameter synthesis deals with establishing the open design parameters associated with preselected candidate isolator elements that satisfy the constraints and cause the performance index to be minimized. Two approaches are possible: (a) direct synthesis, which proceeds from the equations of motion and selects the design parameters in sequential fashion by successively reducing the performance index, and (b) indirect synthesis, which utilizes information gained from first establishing the limiting performance characteristic. While the available results are minimal and considerably less than for shock isolation system design, the solution methods, in some respects, are more straightforward as a consequence of not having to deal in the time domain. We consider both methods in brief.

Direct Synthesis

An example of the direct synthesis of an isolation system for random disturbances is considered in Refs. 10 and 77 and relates to the suspension system of a

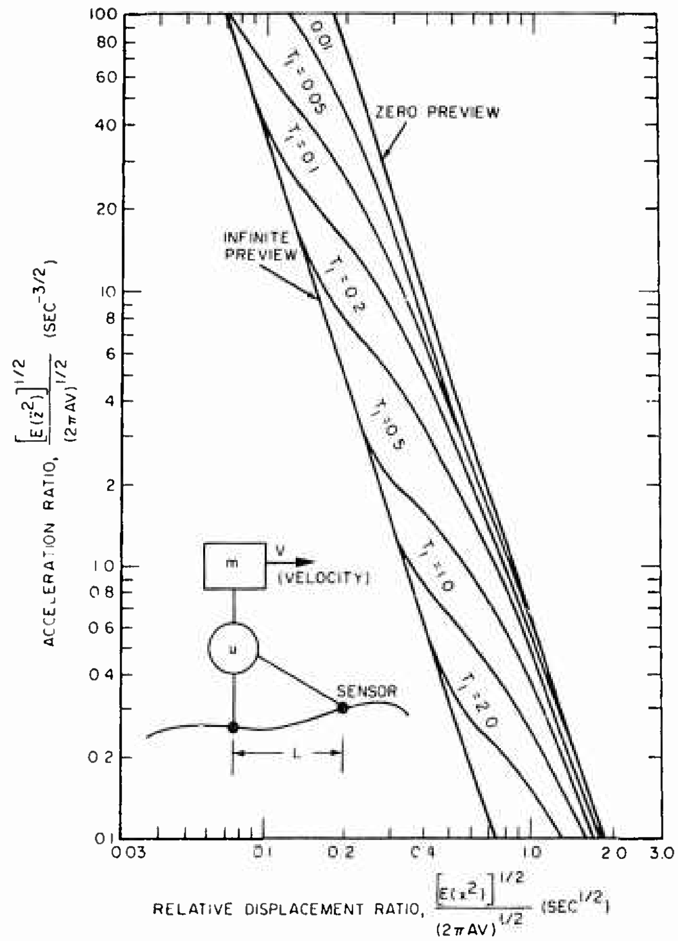


Fig. 8.2. Limiting performance characteristics for system with preview sensing.

vehicle traversing an irregular roadway at constant velocity. The dynamic system is the flexible-base model of Fig. 8.3 with a linear spring and a viscous damper as the candidate isolation element (Fig. 8.5). The disturbance is represented by the power spectral density of Eq. (8.11) and the performance index is of the form of Eq. (8.1). While constraints are not imposed explicitly, the form of Eq. (8.1) has the effect of constraining either the expected mean-square values of the acceleration of m_2 or its relative displacement, depending on whether ρ is small or large. The motion of mass m_1 is entirely unconstrained in this formulation.

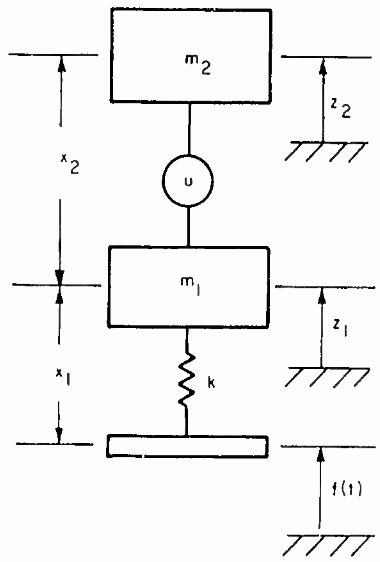


Fig. 8.3. Flexible-base isolation system.

The transfer functions for the pertinent motions of mass m_2 are found to be

$$X_2(\phi) = \frac{\phi^2 F(\phi)}{\phi^4 + 2\zeta\gamma\left(1 + \frac{1}{r}\right)\phi^3 + \left[\gamma^2\left(1 + \frac{1}{r}\right) + 1\right]\phi^2 + 2\zeta\gamma\phi + \gamma^2}$$

$$\ddot{Z}_2(\phi) = \omega^2(2\zeta\gamma\phi + \gamma^2)X_2(\phi), \tag{8.12}$$

where

$$\phi = \frac{s}{\omega}$$

$$F(\phi) = -\frac{AV\omega^2}{\phi^2}$$

$$r = \frac{m_1}{m_2}$$

$$\omega^2 = \frac{k_1}{m_1}$$

$$\gamma^2 = \frac{k_1 m_2}{(k_2 m_1)}$$

$$\zeta^2 = \frac{c^2 k_2 m_2}{4}.$$

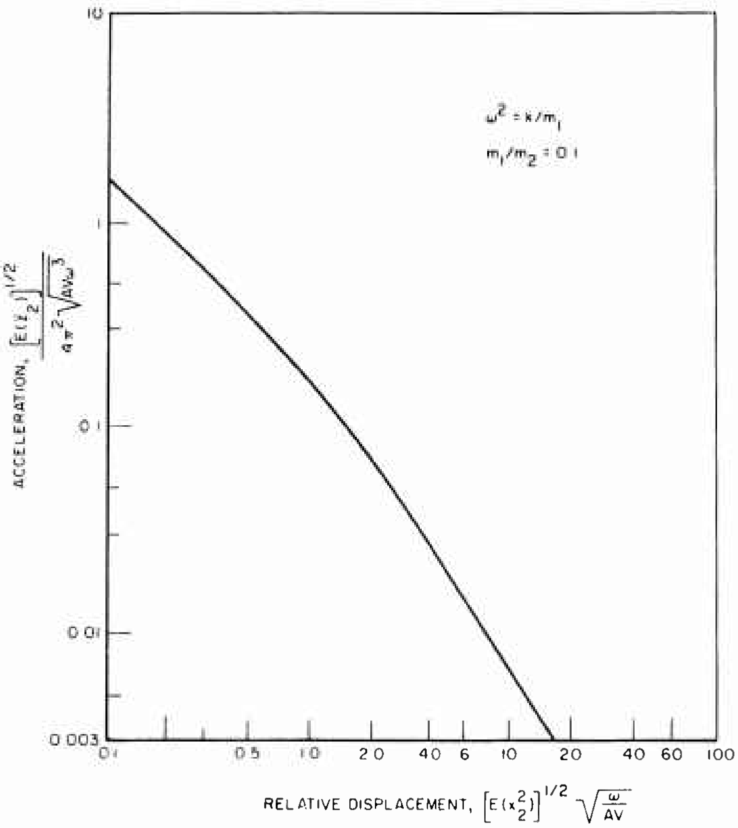


Fig. 8.4. Limiting performance diagram for a flexible-base model.

The expected mean-square values are found from Eq. (8.3); the results are

$$
\begin{aligned}
E(\ddot{z}_2^2) &= 16\pi^4 A V \omega^3 \left[r\zeta\gamma + \frac{(1+r)\gamma^3}{4\zeta} \right] \\
E(x_2^2) &= \frac{(1+r)AV}{4\zeta\gamma\omega} .
\end{aligned}
\tag{8.13}
$$

Those values of ζ and γ that minimize the performance index are readily found. In this case, the optimum performance characteristic is given by

$$
E[\ddot{z}_2^2]\,E[x_2^2] = \pi^4 \omega^2 r(1+r) A^2 V^2 . \tag{8.14}
$$

References 10 and 77 present design charts for the optimum synthesis of this system with the addition of a constant force to mass m_2. This force is included

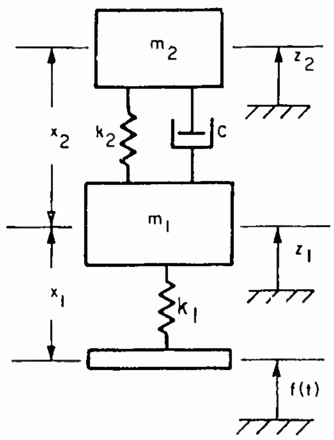


Fig. 8.5. Spring-dashpot isolator with flexible base.

to account for variations in load carried by the suspended body or vehicle (mass m_2). A computational search routine was used to find the parameters k_2 and c for which ψ of Eq. (8.1) is minimized.

Indirect Synthesis

The isolator that achieves the limiting value of the performance index is described by its transfer function $W^*(s)$, e.g., Eq. (8.7). The method of indirect synthesis establishes the design parameters of a selected candidate isolator so as to best approximate the optimum transfer function. An example of this technique is given in Ref. 10, which investigates the active system shown in Fig. 8.6. Accelerometers measure the response of each mass and combine these to form a command signal in an actuator located in parallel with a linear spring and dashpot. The net isolator force in the transform state is

$$U(s) = k_2 x_2 + c s x_2 - (K_{sa}s^2 + K_{sv}s)z_2 + (K_{ua}s^2 + K_{uv}s)z_1 .$$

The transfer function for the acceleration of mass m_2 is found to be

$$\frac{\ddot{Z}(\phi)}{F(\phi)} = \frac{B_1\phi^4 + B_2\phi^3 + B_3\phi^2}{B_4\phi^4 + B_5\phi^3 + B_6\phi^2 + B_7\phi + 1}, \tag{8.15}$$

where

$$B_1 = \frac{\omega^2 r K_{ua}}{\gamma^2 m_1}$$

$$B_2 = \omega^2\left(\frac{2\zeta}{\gamma} + \frac{rK_{uv}}{\gamma^2 m_1 \omega}\right)$$

$$B_3 = \omega^2$$

$$B_4 = \gamma^{-2}\left(1 + \frac{K_{sa}}{m^2} + \frac{K_{ua}}{m_1}\right)$$

$$B_5 = \frac{2\zeta\left(1 + \dfrac{1}{r}\right)}{\gamma} + \frac{K_{sv}}{\gamma^2 m_2 \omega} + \frac{K_{uv}}{\gamma^2 m_1 \omega}$$

$$B_6 = \gamma^{-2} + \frac{K_{sa}}{\gamma^2 m_2} + 1 + \frac{1}{r}$$

$$B_7 = \frac{2\zeta}{\gamma} + \frac{K_{sv}}{\gamma^2 m_2 \omega} .$$

The other quantities are as defined in connection with Eq. (8.12).

The Wiener filter method provides an optimum transfer function for the system of Fig. 8.3, which is of the same form as Eq. (8.15). Here the coefficients

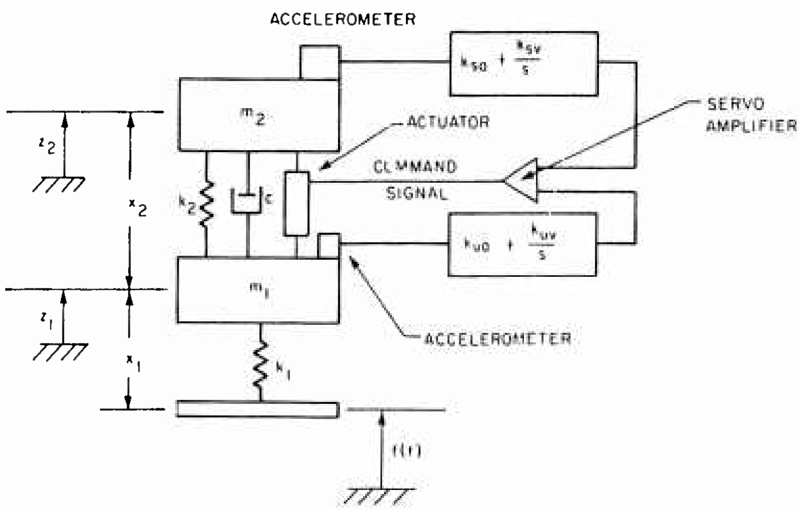


Fig. 8.6. Candidate isolator configuration for flexible-base model.

B_i are expressed in terms of ω, ρ, A, and V. The design parameters k_2, c, K_{sa}, K_{sv}, K_{ua}, and K_{uv} can be selected such that Eq. (8.15) duplicates the optimum transfer function by equating coefficients in like powers of ϕ. In fact, this provides an insufficient number of conditions, and some of the parameters may be selected on the basis of other considerations. For example, K_{uv} can be zero while γ and K_{sa} can be chosen arbitrarily. A chart of optimum design parameters is presented in Ref. 77 for $K_{uv} = K_{sa} = 0$, $r = 0.1$.

Appendix A

GLOSSARY OF SYMBOLS

A	Acceleration constraint
A_f	Characteristic reference acceleration, usually the maximum
a	Amplitude scaling coefficient for input
b	Time scaling coefficient for input
C_k	Constraint functions
c	Viscous damping coefficient
D	Displacement constraint, rattlespace
D_f	Reference displacement, usually the maximum
E[ ]	Expected value of [ ]
f, f_ℓ	Input waveforms, disturbances
f_m	Input disturbance amplitude
g	Acceleration due to gravity
h, h_s	System response quantity
I	Number of discrete time intervals
J	Number of isolator elements
K	Number of constraints
k	Spring constant
L	Number of input waveforms
$\mathcal{L}_n$	Differential operator
M	Number of structural elements
m	Mass
N	Number of position vectors (generalized coordinates)
P	Number of admissible states in dynamic programming solution
q	Kinematic function
R, R_j	Number of isolator design parameters, autocorrelation function
S	Number of response quantities, spectral density
s	Laplace transform variable
t	Time
Δt	Subintervals of time
U_j	Force function for jth isolator
u_j	Force in jth isolator
V	Velocity
w	Initial position coordinate

x	Relative displacement
$\dot{x}$	Relative velocity
z	Acceleration
α, α_{jr}	Design parameters
δ	Dirac delta function
η	Ratio of forcing frequency to natural frequency
λ	Transform parameter
ξ	Coordinate
ρ	Weighting factor, or mass per unit length
σ	Stress
τ	Scaled time or delay time
ϕ	Dynamic programming objective function, or phase angle
ψ	Performance index
ψ_B	Bound to performance index
Ω	Spatial frequency
ω, ω_n	Natural frequency, temporal frequency

Superscripts

$*$	Optimum
L	Lower
U	Upper

Subscripts

F	Index of final time
i	Index of discrete time values
j	Index of isolator element
k	Index of constraints
ℓ	Index of input waveform
m	Maximum
n	Index of system element
p	Index of admissible states
r	Index of design parameters
s	Index of response quantities
0 or 1	Index of initial time

Appendix B

LINEAR PROGRAMMING FORMULATION FOR THE LIMITING PERFORMANCE CHARACTERISTICS OF QUASI-LINEAR SHOCK ISOLATION SYSTEMS

In Chapter 5 the determination of the limiting performance characteristics (i.e., time-optimal synthesis) for quasi-linear systems was shown to be reducible to a problem of linear programming (LP). To utilize existing LP codes, however, it is necessary to obtain a formulation in standard LP terms. The purpose of this appendix is to describe such a formulation. No attempt is made to discuss the solution techniques on which the various existing LP codes are based. The most popular of these is the simplex method, and the reader is referred to the extensive literature on the subject [e.g., 21, 81].

The standard LP problem involves N unknown variables $y_1, y_2, \ldots, y_N$ and NM coefficients, $a_{11}, a_{12}, \ldots, a_{rn}, \ldots, a_{MN}$; M coefficients $b_1, b_2, \ldots, b_r, \ldots, b_M$; and N coefficients, $c_1, c_2, \ldots, c_n, \ldots, c_N$; all of which are known. It is required that $M < N$.

The problem is to find the set of nonnegative y_n which *minimizes* a linear function

$$\sum_{n=1}^{N} c_n y_n \tag{B.1}$$

and satisfies the M equalities

$$\sum_{n=1}^{N} a_{rn} y_n = b_r \qquad \text{for} \quad r = 1, 2, \ldots, M < N. \tag{B.2}$$

Also,

$$y_n \geqslant 0 \qquad \text{for} \quad n = 1, 2, \ldots, N. \tag{B.3}$$

Equations (B.1), (B.2), and (B.3) constitute the standard LP problem in what is known as the *primal formulation.* An alternate, or dual, form admits inequalities in (B.2) and does not impose the nonnegative variable restriction of

(B.3). Specifically, the dual formulation requires that we find the w_r ($r = 1, 2, \ldots, M$) unknown variables that *maximize*

$$\sum_{r=1}^{M} b_r w_r, \tag{B.4}$$

subject to the conditions

$$\sum_{r=1}^{M} a_{rn} w_r \leqslant c_n; \qquad n = 1, 2, \ldots, N. \tag{B.5}$$

Whenever the primal form yields a solution for the y_n, the w_r associated with the dual form are also determined. Moreover, the minimum of Eq. (B.1) is numerically equal to the maximum of Eq. (B.4).

Our purpose is to show how the time-optimal synthesis problem for quasi-linear systems is converted to the standard LP formulation. The time-optimal problem more closely resembles the dual than the primal form because the response constraints are inequalities. However, standard LP codes frequently require the primal form for the input but provide solutions to both the primal and dual problems; i.e., the output includes both y_n and w_r. Solutions to the dual form in such codes are obtained by entering a primal-form input for a dummy problem. For example, the discrete version of our synthesis problem usually is to find the w_r which minimize Eq. (B.4) such that inequalities (B.5) are satisfied. It is clear from the primal and dual formulation statements that this problem can be introduced into an LP computer program as the dummy primal problem by interchanging the rows and columns of the dual and switching the c_n and b_r coefficients.

Inequalities that arise in time-optimal synthesis can be accommodated in the primal form through the introduction of so-called slack variables. Thus, for each inequality of the form

$$\sum_{n=1}^{N} a_{rn} y_n < b_r,$$

we introduce as a slack variable the positive quantity y_{N+1}, defined by the equality condition

$$\sum_{n=1}^{N} a_{rn} y_n + y_{N+1} = b_r. \tag{B.6}$$

Similarly, for an inequality of the form

$$\sum_{n=1}^{N} a_{rn} y_n > b_r,$$

a positive slack variable y_{N+1} is defined so that

$$\sum_{n=1}^{N} a_{rn} y_n - y_{N+1} = b_r. \tag{B.7}$$

If we wish to admit the possibility of some of the unknown y_n being negative, i.e., if Eq. (B.3) does not hold for all n, we may represent y_n as the difference of two nonnegative variables, i.e.,

$$y_n = y_n' - y_n''; \qquad y_n' \geqslant 0, \ y_n'' \geqslant 0. \tag{B.8}$$

We see, therefore, that the standard primal formulation can be generalized at the expense of introducing additional unknown variables.

It is convenient to represent the LP formulation in matrix notation. Our purpose will be to show how these matrices are evaluated in terms of the parameters of the time-optimal synthesis problem using the notation of Chapters 4 and 5. Equations (B.1), (B.2), and (B.3) are written as

$$\begin{aligned} \psi &= \mathbf{c}^T \mathbf{y} \\ \mathbf{A}\mathbf{y} &= \mathbf{b} \\ \mathbf{y} &= 0, \end{aligned} \tag{B.9}$$

where ψ is the performance index to be minimized; superscript T indicates the transpose; $\mathbf{y}$ is the vector of unknown variables $\{y_n\}$; $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{c}$ are, respectively, the matrix $[a_{rn}]$ and the vectors $\{b_r\}$ and $\{c_n\}$ of known coefficients; and it is understood that the rank of these elements, M, N, is expanded to include the necessary number of slack variables. The condition that $M < N$ is necessary since if $M > N$ there would be $M - N$ redundant equations, which could be eliminated, or if $M = N$ there would be either a unique solution or no solution, depending on the consistency of the constraints. The requirement that $M < N$ ensures the existence of an infinite number of solutions to Eq. (B.9) among which we seek the one (or ones) that minimizes ψ.

The time-optimal synthesis problem requires us to find the J isolator force functions $u_j(t)$ that minimize the performance index (see Eq. (4.6))

$$\psi = \max_s \max_t |h_s(t, u_j)|; \qquad s = 1, 2, \ldots, S,$$

subject to . .. K constraints (see Eq. (4.7))

$$C_k^L(t) \leqslant C_k(t, u_j) \leqslant C_k^U(t); \qquad k = 1, 2, \ldots, K.$$

Note that minimization of ψ as a max-max form is equivalent to minimizing ψ subject to the constraint

$$|h_s(t, u_j)| \leqslant \psi \qquad \text{for all } t \text{ and } s = 1, 2, \ldots, S.$$

The function h_s is the response quantity of interest and is evaluated from the solution to the system dynamics, as are the constraint functions C_k. For simplicity of notation we will discuss only the case where the performance index depends on a single response quantity, i.e., S=1, and we will drop the subscript s. The case of S>1 introduces additional constraint relations in an obvious manner.

The continuous functions are replaced by discrete quantities evaluated at the times t_i as described in Section 5.1.1. Each isolator force function is represented by the vector with elements $\{u_{ji}\}$ for $i = 1, 2, \ldots, I$. The discrete version of the time-optimal synthesis problem thus is to find the u_{ji}, $j = 1, 2, \ldots, J$; $i = 1, 2, \ldots, I$ so that ψ is minimized and the constraints

$$\begin{aligned} C_{ki}^L \leqslant C_k(t_i, u_{ji}) &= C_{ki} \leqslant C_{ki}^U \\ |h(t_i, u_{ji})| &= |h_i| \leqslant \psi \end{aligned} \tag{B.10}$$

or, equivalently

$$\begin{aligned} C_{ki} - C_{ki}^U &\leqslant 0 \\ C_{ki} - C_{ki}^L &\geqslant 0 \\ h_i - \psi &\leqslant 0 \\ \psi + h_i &\geqslant 0, \end{aligned} \tag{B.11}$$

are satisfied for $k = 1, 2, \ldots, K$ and $i = 1, 2, \ldots, I$.

We consider three forms for the response function and two forms for the constraint function as follows. These include most cases of practical interest:

$$h(t, u_j) = \begin{cases} h_0(t) + \sum_{j=1}^{J} \int_0^t R_j(t-\tau)u_j(\tau)d\tau & \text{(a)} \\ \sum_{j=1}^{J} q_j u_j(t) & \text{(b)} \\ u_\ell(t); \quad \ell = 1 \text{ or } 2 \text{ or } \ldots \text{ or } J & \text{(c)} \end{cases} \tag{B.12}$$

$$C_k(t, u_j) = \begin{cases} C_{k0}(t) = \sum_{j=1}^{J} \int_0^t R_{kj}(t-\tau)u_j(\tau)d\tau & \text{(a)} \\ C_{k0}(t) + \sum_{j=1}^{J} q_{kj}u_j(t) & \text{(b)} \end{cases} \tag{B.13}$$

As explained in Section 5.1.2, $h_0(t)$ and $C_{k0}(t)$ are the system responses of interest to initial conditions and input disturbances, whereas $R_j(t)$ and $R_{kj}(t)$ are the system responses to a unit impulse applied at the attachment points of the jth isolator; q_j and q_{kj} are prescribed constants. On occasion the time t in the constraint functions (B.13) assumes simply a terminal value; for example, the time at which the input has decayed to zero.

The forms of Eq. (B.12) and (B.13) depend on the discrete approximations adopted for $u_j(t)$. We will consider both piecewise constant and piecewise linear approximations as shown in Fig. B.1. Let

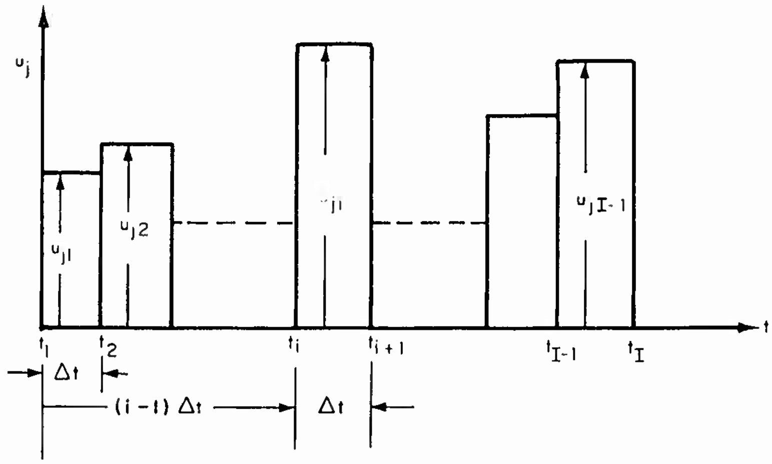
$$g_j(t) = \int_0^t R_j(t-\tau)u_j(\tau)d\tau. \tag{B.14}$$

In discrete form, g_{ji} can be represented as a vector,

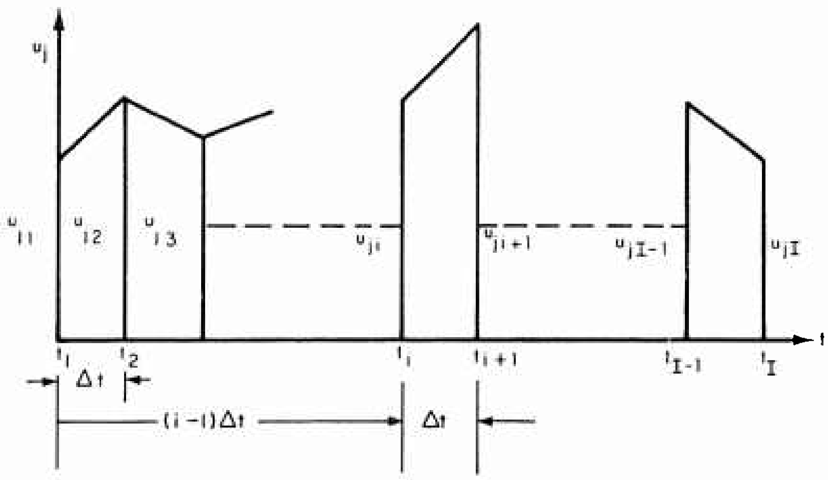
$$\mathbf{g}_j = \mathbf{D}_j \cdot \mathbf{u}_j, \tag{B.15}$$

where the matrix $\mathbf{D}_j$ depends on the approximation for $\mathbf{u}_j$. It is easily shown that the elements $\{d_{jiq}\}$ of the $\mathbf{D}_j$ matrix are, for the piecewise constant approximation with $I-1$ components of $u_j(t)$,

$$d_{jiq} = \begin{cases} \frac{\Delta t}{2}(R_{ji-q} - R_{ji-q+1}) & \text{for } q = 1, \ldots, i-1 \\ 0 & \text{for } q = i, \ldots, I-1 \end{cases} \tag{B.16}$$



(a) Piecewise constant



(b) Piecewise linear

Fig. B.1. Discretization of the isolator force.

and, for the piecewise linear approximation,

$$d_{jiq} = \frac{\Delta t}{6}\begin{cases} (R_{ji-1} + 2R_{ji}) & q = 1\ (\neq i) \\ (R_{ji-q} + 4R_{ji-q+1} + R_{ji-q+2}) & q = 2, \ldots, i-1 \\ (2R_{j1} + R_{j2}) & q = i \\ 0 & q = i+1, \ldots, I. \end{cases} \tag{B.17}$$

The integrals involved in $C_k(t, u_j)$ are discretized in a similar fashion using matrix, say, D_{jk}. The elements d_{jiq} for the response and constraint functions not represented with integrals, Eq. (B.12b and c) and Eqs. (B.13b) can be identified by observation.

The time-optimal synthesis problem as formed from Eqs. (B.11), (B.12), (B.13), and (B.15) is to find ψ and u_{ji} such that ψ is minimized and the constraints

$$\begin{aligned} \sum_{j=1}^{J} \mathrm{D}_{jk} \cdot \mathrm{u}_j &\leqslant -C_{k0}(t_i) + C_{ki}^U \\ -\sum_{j=1}^{J} \mathrm{D}_{jk} \cdot \mathrm{u}_j &\leqslant C_{k0}(t_i) - C_{ki}^L \\ -\psi + \sum_{j=1}^{J} \mathrm{D}_j \cdot \mathrm{u}_j &\leqslant -h_0(t_i) \\ -\psi - \sum_{j=1}^{J} \mathrm{D}_j \cdot \mathrm{u}_j &\leqslant h_0(t_i) \end{aligned} \tag{B.18}$$

are satisfied for all i and k. This is now in the dual LP form with $\{b_r\} = \{1, 0, 0, \ldots, 0\}$ and w_n, a_{rn}, and c_n developed from Eq. (B.18) as shown in Table B.1. This can be entered into an LP computer program directly, if acceptable to the code, or as a dummy primal problem. The matrices of Table B.1 are intended to be representative of a typical time-optimal synthesis problem. They may require minor adjustments to accommodate special problem statements.

Consider the direct formation of the primal LP problem. The performance index ψ is necessarily nonnegative. The u_{ji} are made positive in the manner of Eq. (B.8), i.e., $u_{ji} = u'_{ji} - u''_{ji}$, $u'_{ji} \geqslant 0, u''_{ji} \geqslant 0$. Relations (B.18) are converted to equality constraints using positive quantities h'_i, h''_i, C'_{ki}, C''_{ki}. Thus, we now seek $\psi, u'_{ji}, u''_{ji}, h'_i, h''_i, C'_{ki}, C''_{ki}$ such that ψ is minimized subject to

Table B1. Matrices for Dual Linear Programming Formulation

Notation: $\psi = \mathbf{b}^T\mathbf{w}$, constraints $\mathbf{A}^T\mathbf{w} \leqslant \mathbf{c}$: $N_u = \begin{cases} I-1 & \text{for piecewise constant } u_j \\ I & \text{for piecewise linear } u_j \end{cases}$

$$
\begin{array}{cc}
 & \begin{array}{ccccc} 1 & N_u & N_u & & N_u \end{array} \\
\mathbf{w} = & \left\{ \begin{array}{c|c|c|c|c} \psi & \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_J \end{array} \right\}
\end{array}
$$

$$
\mathbf{A}^T = \left[\begin{array}{c|c|c|c|c}
0 & \mathbf{D}_{11} & \mathbf{D}_{21} & \cdots & \mathbf{D}_{J1} \\
\hline
\multicolumn{5}{c}{\text{Continue on } k \downarrow} \\
\hline
0 & \mathbf{D}_{1K} & \mathbf{D}_{2K} & \cdots & \mathbf{D}_{JK} \\
\hline
0 & -\mathbf{D}_{11} & -\mathbf{D}_{21} & \cdots & -\mathbf{D}_{J1} \\
\hline
\multicolumn{5}{c}{\text{Continue on } k \downarrow} \\
\hline
0 & -\mathbf{D}_{1K} & -\mathbf{D}_{2K} & \cdots & -\mathbf{D}_{JK} \\
\hline
-1 & \mathbf{D}_1 & \mathbf{D}_2 & \cdots & \mathbf{D}_J \\
\hline
-1 & -\mathbf{D}_1 & -\mathbf{D}_2 & \cdots & -\mathbf{D}_J
\end{array} \right]
\qquad
\mathbf{c} = \left[\begin{array}{c}
-C_{10}(t_i) + C_{1i}^U \\
\hline
\cdots \\
\hline
-C_{K0}(t_i) + C_{Ki}^U \\
\hline
C_{10}(t_i) - C_{1i}^L \\
\hline
\cdots \\
\hline
C_{K0}(t_i) - C_{Ki}^L \\
\hline
-h_0(t_i) \\
\hline
h_0(t_i)
\end{array} \right]
\begin{array}{l}
N_u \\ \\ N_u \\ N_u \\ \\ N_u \\ N_u \\ N_u
\end{array}
\begin{array}{l}
\\ K \cdot N_u \\ \\ \\ K \cdot N_u \\ \\ \\ \\
\end{array}
\begin{array}{l}
\\ \\ \\ 2(K+1) \cdot N_u \\ \\ \\ \\ \\
\end{array}
$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{1 + J \cdot N_u}$$

$$\begin{aligned}
\sum_{j=1}^{J} \mathbf{D}_{jk} \cdot (\mathbf{u}_j' - \mathbf{u}_j'') + C_{ki}' &= -C_{k0}(t_i) + C_{ki}^{U} \\
-\sum_{j=1}^{J} \mathbf{D}_{jk} \cdot (\mathbf{u}_j' - \mathbf{u}_j'') + C_{ki}'' &= C_{k0}(t_i) - C_{ki}^{L} \\
-\psi + \sum_{j=1}^{J} \mathbf{D}_j \cdot (\mathbf{u}_j' - \mathbf{u}_j'') + h_i' &= -h_0(t_i) \\
-\psi - \sum_{j=1}^{J} \mathbf{D}_j \cdot (\mathbf{u}_j' - \mathbf{u}_j'') + h_i'' &= h_0(t_i)
\end{aligned} \tag{B.19}$$

for all i, k. The vector of unknowns $\mathbf{y}$ now appears as

$$\begin{aligned}
\mathbf{y}^T = \{ &\psi; u_{11}', \ldots, u_{1I}', u_{21}', \ldots, u_{JI}'; u_{11}'', \ldots, u_{1I}'', u_{21}'', \ldots, u_{JI}''; \\
&h_1', \ldots, h_I'; h_1'', \ldots, h_I''; C_{11}', \ldots, C_{1I}', C_{21}', \ldots, C_{KI}'; \\
&C_{11}'', \ldots, C_{1I}''; C_{21}'', \ldots, C_{KI}'' \}.
\end{aligned}$$

Equations (B.19) are easily placed in tabular form similar to that of Table B1. Reference 3 contains details of this formulation including discussions of reduction in matrix sizes possible for special cases. For example, $\mathbf{y}$ need not contain u_{ji}'' if u_{ji} is bounded from below.

REFERENCES

1. D. C. Karnopp and A. K. Trikha, "Comparative Study of Optimization Techniques for Shock and Vibration Isolation," *J. Eng. Ind.*, **91**, 1128-1132 (1969).
2. C. D. Johnson, "Study of Optimal and Adaptive Control Theory," NASA CR-715, Apr. 1967.
3. W. D. Pilkey, E. H. Fey, T. Liber, A. J. Kalinowski, and J. P. Costello, "Shock Isolation Synthesis Study," SAMSO TR 68-388, Vol. I and II, Feb. 1969, Space and Missile Systems Organization, USAF, San Bernardino, Calif.
4. H. W. Kriebel, "A Study of the Feasibility of Active Shock Isolation," *Ing. Arch.*, **36** (6), 371-380 (1968).
5. H. Kriebel, "A Study of the Practicality of Active Shock Isolation," unpublished Ph.D. dissertation, Stanford Univ., May 1966.
6. R. Bellman, "On Minimizing the Probability of a Maximum Deviation," *IRE Trans. Automat. Contr.*, **7** (2), 45 (1962).
7. A. K. Trikha and D. C. Karnopp, "A New Criterion for Optimizing Linear Vibration Isolator Systems Subject to Random Input," *J. Eng. Ind.*, **91**, 1005-1010 (1969).
8. J. E. Ruzicka, "Characterization of Mechanical Vibration and Shock," *Sound and Vibration*, **1** (4), 14-21 (1967).
9. R. V. Churchill, *Fourier Series and Boundary Value Problems*, McGraw-Hill Book Co., New York, 1941.
10. E. K. Bender, D. C. Karnopp, and I. L. Paul, "On the Optimization of Vehicle Suspensions Using Random Process Theory," ASME Paper 67-TRAN-12, *Mech. Eng.*, **89**, 69 (1967).
11. W. E. Thompson, "Measurements and Power Spectra of Runway Roughness at Airports in Countries of the North Atlantic Treaty Organization," NACA TN-4304, July 1958.
12. A. Craggs, "The Assessment of Pave Test Track Loads Using Random Vibration Analysis," in *Advances in Automobile Engineering* (J. H. Tidbury, editor) Pergamon Press, London, Vol. 7, p. 43, 1965.
13. A. O. Gilchrist, "A Report on Some Power-Spectral Measurements of Vertical Rail Irregularities," British Railway Research Department, Derby, England (1965).
14. D. C. Karnopp, "Applications of Random Process Theory to the Design and Testing of Ground Vehicles," *Transp. Res.*, **2**, 269-278 (1968).

15. T. Liber, "Optimum Shock Isolation Synthesis," AFWL-TR-65-82, Air Force Weapons Laboratory, Albuquerque, N. Mex., July 1966.
16. R. A. Eubanks, T. Liber, W. D. Pilkey, and R. L. Barnett, "Optimal Shock Isolator and Absorber Design Techniques," IIT Research Institute Final Report on ONR Contract Nonr-444-4(00)X (Apr. 1965).
17. W. D. Pilkey, "Optimum Mechanical Design Synthesis," Vol. I and II, IITRI Final Report M6089 (June 1966) Contract No. DA-31-124-ARO-D-243.
18. R. A. Eubanks, "Investigation of a Rational Approach to Shock Isolator Design," *Shock and Vibration Bull.*, **39**, Pt. 3, 157-168 (1964).
19. K. T. Cornelius, "Rational Shock Mount Design, Investigation of Efficiency of Damped, Resilient Mount," Naval Ship Research and Development Center Report 2383 (July 1967).
20. L. A. Schmit, "Structural Synthesis," summer course notes, Case Institute of Technology, Cleveland, Ohio, July 1966.
21. N. P. Loomba, *Linear Programming*, McGraw-Hill Book Co., New York, 1964.
22. R. Bellman and S. Dreyfus, *Applied Dynamic Programming*, Princeton University Press, Princeton, N.J., 1962.
23. B. Porter, "Synthesis of Optimal Suspension Systems," *Engineer*, **223**(5805), 619-622 (1957).
24. J. Wolkovich, "Techniques for Optimizing the Response of Mechanical Systems to Shock and Vibration," SAE Paper 680748 (1968).
25. D. C. Karnopp, "Random Search Techniques for Optimization Problem," *Automatica*, **1**, 111-121 (1963).
26. B. D. Tapley and J. M. Lewallen, "Comparison of Several Numerical Optimization Methods," *J. Optimization Theory Appl.*, **1** (1), 1-32 (1967).
27. H. A. Spang, "A Review of Minimization Techniques for Nonlinear Functions," *SIAM Rev.*, **4** (4), 343-365 (1962).
28. A. Zoulendijk, "Nonlinear Programming: A Numerical Survey," *J. SIAM Contr.*, **4** (1), 194-210 (1966).
29. J. Kowalik, "Nonlinear Programming Procedures and Design Optimization," *Acta Polytech. Scand., Math. Comput. Mach.*, Ser. 13 (1966).
30. D. S. Hague and C. R. Glatt, "An Introduction to Multivariable Search Techniques for Parameter Optimization," (and Program AESOP) NASA CR-73200 (1968).
31. M. J. Box, "A Comparison of General Optimization Methods and the Use of Transformation in Constraint Problems," *Comput. J.*, **9**, 67-77 (1966).
32. S. H. Brooks, "A Comparison of Maximum-Seeking Methods, *Oper. Res.*, **7**, 430-457 (1959).
33. B. Carnahan and J. O. Wilkes, "Numerical Methods, Optimization Techniques, and Simulation for Engineers," University of Michigan, Engineering Summer Conferences, An Intensive Short Course, May 20-31 (1968).
34. A. F. Fiacco and G. P. McCormick, *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*, John Wiley, New York, 1968.

35. A. V. Murty and G. V. Murty, "Design Modification in Vibration Problems," *J. Spacecraft and Rockets*, 5 (7), 370-372 (1968).
36. J. B. Weissenburger, "Effect of Local Modifications on the Vibration Characteristics of Linear Systems," *J. Appl. Mech.*, **35** (2), 327-332 (1968).
37. F. J. Melosh and R. Luk, "Approximate Multiple Configuration Analysis and Allocation of Least Weight Structural Design," AFFDL-67-59 (Apr. 1969), Air Force Fluid Dynamics Laboratory, Wright-Patterson AFB, Ohio.
38. L. A. Schmit and E. F. Rybicki, "Simple Shock Isolator Synthesis with Bilinear Stiffness and Variable Damping," NASA CR-64-710 (June 1965).
39. L. A. Schmit and R. Fox, "Synthesis of a Simple Shock Isolator," NASA CR-55 (June 1964).
40. E. Davidson and K. Cowan, "Functional Optimization by Rosenbrock's Direct Method," SHARE Program Library, SDA 5466 (1966), IBM Corporation.
41. I. L. Paul and E. K. Bender, "Active Vibration Isolation and Active Vehicle Suspension," MIT Dept. of Mechanical Engineering, Nov. 1966 (PB173,648).
42. J. Ormondroyd and J. P. Den Hartog, "Theory of the Dynamic Vibration Absorber," *Trans. ASME*, **50** (1928).
43. E. Hahnkamm, "Die Dampfung von Fundament-schwingungen bei veranderlicher Erregerfrequenz," *Ing. Arch.*, **4** (1933).
44. J. E. Brock, "A Note on the Damped Vibration Absorber," *J. Appl. Mech.*, **13**, A284 (1946).
45. B. E. O'Connor, "The Viscous Torsional Vibration Damper," *S.A.E. Quart. Trans.*, **1** (1947).
46. J. E. Brock, "Theory of the Damped Dynamic Vibration Absorber for Inertial Disturbances," *J. Appl. Mech.*, **16**, A86 (1949).
47. F. M. Sauer and C. F. Garland, "Performance of the Viscously Damped Vibration Absorber Applied to Systems Having Frequency Squared Excitation," *J. Appl. Mech.*, **16**, A109 (1949).
48. R. E. Roberson, "Synthesis of a Non-Linear Dynamic Vibration Absorber," *J. Franklin Inst.*, **254**, 205 (1952).
49. F. R. Arnold, "Steady-State Behavior of Systems Provided with Non-Linear Dynamic Vibration Absorbers," *J. Appl. Mech.*, 22, 487 (1955).
50. F. M. Lewis, "Extended Theory of Viscous Vibration Damper," *J. Appl. Mech.*, 22, 377 (1955).
51. V. A. Radzievskii, "Problem of the Errors and Optimum Damping in the Single Component Vibration Measuring Apparatus of the Seismic Type," *Dopov. Akad. Nauk UkrSSR*, 5, 426 (1956).
52. R. Plunkett, "The Calculation of Optimum Concentration, Damping for Continuous Systems," *J. Appl. Mech.*, 25, 219 (1958).
53. E. M. Kerwin, Jr., "Damping of Flexural Waves by a Constrained Visco-Elastic Layer," *J. Acoust. Soc. Amer.*, **31** (7), 952 (1959).
54. W. J. Carter and F. C. Liu, "Steady-State Behavior of Non-Linear Dynamic Vibration Absorber," *J. Appl. Mech.*, 28, 67 (1961).

55. J. E. Ruzicka, "Damping Structural Responses Using Viscoelastic Shear-Damping Mechanisms," *J. Eng. Ind.*, **83B**, 403 (1961).
56. J. F. Springfield and J. P. Raney, "Experimental Investigation of Optimum End Supports for a Vibrating Beam," *Exp. Mech.*, 2 (12), 366 (1962).
57. D. G. Bogy and P. R. Paslay, "Evaluation of Fixed Point Method of Vibration Analysis for a Particular System with Initial Damping," *J. Eng. Ind.*, **85B**, 233 (1963).
58. A. R. Henny and J. P. Raney, "The Optimization of Damping of Four Configurations of a Vibrating Beam," *J. Eng. Ind.*, **85B**, 259 (1963).
59. R. Plunkett, "Vibration Response of Linear Damped Complex Systems," *J. Appl. Mech.*, **30**, 70 (1963).
60. A. R. Henny, "The Damping of Continuous Systems," *Engineer*, **215** (5529), 572 (1963).
61. T. J. Mentel, "Viscoelastic Boundary Damping of Beams and Plates," *J. Appl. Mech.*, **31**, 61 (1964).
62. D. J. Mead, "The Damping of Beam Vibration by Rotational Damping at the Supports," Institute of Sound and Vibration Research Rpt. 121, Southampton, England, 1964.
63. V. H. Neubert, "Dynamic Absorbers Applied to Bars that have Solid Damping," *J. Acoust. Soc. Amer.*, **36** (4), 673 (1964).
64. P. Vausherk and J. Peters, "Optimization of Dynamic Shock Absorption for Machinery," *CIRP Ann.*, **12** (2), 120-126 (1963).
65. R. Plunkett and C. H. Wu, "Attenuation of Plane Waves in Semi-Infinite Composite Bar," *J. Acoust. Soc. Amer.*, **37** (1), 28 (1965).
66. J. C. Snowdon, "Vibration of Cantilever Beams to Which Dynamic Absorbers are Attached," *J. Acoust. Soc. Amer.*, **39** (5), 878 (1966).
67. F. R. Gekker, "Determination of the Parameters of 'Dry Friction' Damper," *Izv. Vyssh. Ucheb. Zaved. Mashinostr.*, No. 3, pp. 53-56, 1966 (in Russian).
68. J. A. Bonesho and J. G. Bollinger, "Theory and Design of a Self-Optimizing Damper," pp. 229-241, *Proc., 7th Internat'l Machine Tool Design and Res. Conf.*, Univ. of Birmingham, England (Sept. 1966).
69. J. E. Ruzicka, T. F. Derby, D. W. Schubert, and J. S. Pepi, "Damping of Structural Composites with Viscoelastic Shear-Damping Mechanisms," NASA CR-742, (1967).
70. D. I. G. Jones, "Response and Damping of a Single Beam with Tuned Absorbers," *J. Acoust. Soc. Amer.*, **42** (1), 50 (1967).
71. K. A. Kriukov, *Determination of the Parameters of an Elastically Damping Turbine-Rotor Bearing*, Izdatel'stvo Mashinostroenie, Moscow, pp. 22-49, 1967.
72. K. C. Falcon, "Optimization of Vibration Absorbers: A Graphical Method for Use on Idealized Systems with Restricted Damping," *J. Mech. Eng. Sci.*, **9** (5), 374 (1967).

73. B. F. Stone, "Optimization of Vibration Absorbers: Iterative Methods for Use on Systems, with Experimentally Determined Characteristics," *J. Mech. Eng. Sci.*, **9** (5), 382 (1967).
74. J. A. Bonesho and J. S. Bollinger, "How to Design a Self-Optimizing Vibration Damper," *Mach. Des.*, **40** (5), 123 (1968).
75. J. C. McMunn, "Multi-Parameter Optimum Damping in Linear Dynamical Systems," unpublished Ph.D. dissertation, University of Minn., 1967.
76. J. C. McMunn and R. Plunkett, "Multi-Parameter Optimum Damping in Linear Dynamical Systems," Am. Soc. Mech. Engs. Vibrations Conf., Paper 69-VIBR-42.
77. E. K. Bender, "Optimization of the Random Vibration Characteristics of Vehicle Suspensions," unpublished Ph.D. dissertation, M.I.T., June 1967.
78. G. C. Newton, L. A. Gould, and J. F. Kaiser, *Analytical Design of Linear Feedback Controls,* John Wiley and Sons, New York, 1957.
79. R. Magdaleno and J. Wolkovitch, "Performance Criteria for Linear Constant-Coefficient Systems with Random Inputs," ASD-TDR-62-470, 1963.
80. E. K. Bender, "Optimum Linear Preview Control with Application to Vehicle Suspension," ASME Paper 67-WA/Aut-1, *J. Basic Eng.*, **90** (2) 213-221 (1968).
81. S. I. Gass, *Linear Programming,* McGraw-Hill Book Co., New York, 1964.

ANNOTATED BIBLIOGRAPHY

SHOCK ISOLATION SYSTEMS

1. R. E. Blake, "Near-Optimum Shock Mounts for Protecting Equipment from an Acceleration Pulse," *Shock and Vibration Bull. No. 35*, 133-146 (Feb. 1966).

   This paper studies the time-optimal response of single isolator systems with flexible equipment. An approximate linear elastic representation is provided for the equipment and a time-consuming nonlinear programming technique is used to carry out the optimization.

2. K. T. Cornelius, "Rational Shock Mount Design. Investigation of Efficiency of Damped, Resilient Mount," Naval Ship Research and Development Center, Report 2383, July 1967.

   The graphical derivation and characteristics of time-optimal synthesis of a single-degree-of-freedom shock isolation system and the possibility of modeling these characteristics with a linear spring in parallel with various damping devices are studied. Typical environments for a ship subject to underwater explosion attack are used as input motions.

3. K. T. Cornelius, "A Study of the Performance of an Optimum Shock Mount," *Shock and Vibration Bull. No. 38*, Pt. 3, 213-219 (Nov. 1968).

   A single-degree-of-freedom shock isolator comprising a linear spring in parallel with a bilinear damper, which is proportional to velocity at low levels and assumes a constant value at higher velocity, is considered. Response of this isolator is compared for several input motions to the time-optimal characteristics of a single-degree-of-freedom system. The idea for this type of isolator configuration grew from a study of properties of the ideal (time-optimal) mount.

4. T. F. Derby and P. C. Calcaterra, "Response and Optimization of an Isolation System with Relaxation Type Damping," *Shock and Vibration Bull., No. 40* (1970).

   The authors consider relaxation-type damping to be an isolator element composed of either a Voigt viscoelastic model in series with an elastic spring, or a standard linear solid viscoelastic model. Inputs are impulse and white noise acceleration of the base. An analytical direct optimal synthesis study is performed for a single-degree-of-freedom system on the basis of the type of acceleration and rattlespace criteria formulated in Chapters 5, 6, and 7 of the monograph. Peak acceleration-vs-rattlespace

tradeoffs for elements with optimum parameters are plotted as dimensionless design curves and compared with the limiting performance characteristics. This is a thorough study of the problem posed.

5. R. A. Eubanks, "Investigation of a Rational Approach to Shock Isolator Design," *Shock and Vibration Bull. No. 34*, Pt. 3, 157-168 (Dec. 1964).

   Techniques possibly suitable for the optimum synthesis of shock isolation design are surveyed. A mathematical statement of optimum shock isolator design problems is given along with anticipated complications in achieving a solution. As documented in this monograph, many of these projected difficulties have been overcome in more recent work.

6. H. E. Gollwitzer, "Rocket Booster Control," Sect. 16, *A Minimax Control for a Plant Subjected to a Known Load Disturbance*, Minneapolis-Honeywell MPG Report 1541-TR-16, Minneapolis, (Jan. 1964).

   This report includes a very complete description of the min-max control problem that is equivalent to the multiple-isolator, multiple-degree-of-freedom time-optimal shock isolation problem formulation of Chapter 5 for prescribed inputs.

7. V. V. Guretskii, "On the Optimization of Shock Isolators," *Tr. Leningrad. Politekh. Inst.*, 252, 16-23 (1965) (in Russian).

   This paper contains a formulation for the time-optimal synthesis of a multiple-isolator, multiple-degree-of-freedom shock isolation system, but does not provide complete solutions. Peak relative displacements throughout the system are to be minimized, whereas peak accelerations are bounded. Problem formulation is similar to that given for the same problem in Chapter 5 of the monograph.

8. V. V. Guretskii, "On a Certain Optimal Control Problem," *Izv. Akad. Nauk SSSR, Mekh.*, **1** (1), 159-162 (Jan.-Feb. 1965) (in Russian).

   The time-optimal problem for a single-degree-of-freedom shock isolation system with a bounded peak acceleration and rattlespace as a performance index is treated. This is the same problem that is given major consideration in the monograph and solution is similar to the graphical technique described in Chapter 5. Solutions are developed for stepped, exponentially decaying, and quarter-circle acceleration disturbances.

9. V. V. Guretskii. "Selection of Optimum Design Parameters for Shock Isolators," *Mekh. Tverd. Tela.*, 1, 167-170 (1966) (in Russian).

   This paper attempts to analytically deduce the optimum design parameters for given simple linear configurations of a single-degree-of-freedom shock isolator. Parameters that lead to a minimal rattlespace while satisfying a bound on peak mass acceleration are sought. The technique does not appear suitable for application to more complicated higher order systems.

10. D. C. Karnopp and A. K. Trikha, "Comparative Study of Optimization Techniques for Shock and Vibration Isolation," AFOSR-68-0242, Air Force Office of Scientific Research, Arlington, Va. (Jan 1968).

Several a· lute optimum and near optimum isolation systems are considered w.. i respect to min-max, quadratic, and expected mean-square value criteria. It is shown that the systems designed on the basis of one criterion do not necessarily respond favorably with respect to other criteria. The report version contains several important appendixes not included in the paper.

11. H. W. Kriebel, "A Study of the Practicality of Ac ve Shock Isolation," unpublished Ph.D. dissertation, Stanford Univ. (May 1966).

This study is similar to that of item 14, with the additional criterion that response variables return to their initial state in minimum time. This represents an effort to uniquely specify the isolator force over the total time interval of interest. A comparative study is made using an integral of the square of the absolute acceleration as a performance index.

12. T.N.T. Lack and M. Enns, "Optimal Control Trajectories with Minimax Objective Functions by Linear Programming," *IEEE Trans. Automat. Contr.*, AC-12 (6), 749-752 (1967).

The formulation and development of a linear programming preprocessing code that places the type of problem found in the optimization of multiple-isolator, multiple-degree-of-freedom isolation systems with linear structural elements in standard linear programming form are described. This code, which is similar to that developed in item 19 and discussed in Appendix B of the monograph, accepts a wide variety of constraints and a maximum in time response as a performance index.

13. Chong Won Lee, "Minimization of the Maximum Value of Cost Function," unpublished Master's thesis, Illinois Institute of Technology, June 1967.

An analytical study of various aspects of control systems with min-max criteria is carried out. The formulation is equivalent to the optimum shock isolation problem if initial and terminal conditions and no external excitation are admitted.

14. T. Liber and E. Sevin, "Optimal Shock Isolation Synthesis," *Shock and Vibration Bull. No. 35*, Pt. 5, 203-215 (Feb. 1966).

This study of the time-optimal synthesis of a single-degree-of-freedom shock isolation system contains linear and dynamic programming formulations of the problem of minimizing rattlespace for a prescribed level of acceleration attenuation. These are precursors to the formulations presented in the monograph.

15. T. Liber, "Optimal Shock Isolation Synthesis," AFWL-TR-65-82, (July 1966), Air Force Weapons Laboratory, Albuquerque, N. Mex.

This expanded version of the Liber-Sevin paper of the same title includes optimum performance characteristics (termed *trade-off limit curves* in the report) for linear isolation systems with steady-state harmonic disturbances.

16. G. G. Love and A. Lavi, "Evaluation of Feedback Structures," Proc. *Joint Automat. Contr. Conf. 1968*, University of Michigan.

    This paper describes a procedure somewhat akin to indirect synthesis, although not computationally oriented, and suggests that rapid evaluation of control system configurations (i.e., candidate forms of systems in terms of state variables) is possible by substituting the trajectories of the desired response in place of the real response as a means of finding optimum design parameters. The procedure is illustrated with an integral form of performance index and avoids repetitive analysis of the system equations of motion. Desired response can be selected on the basis of available information, including the optimum response characteristics. The design parameters found for the best configuration are then used as the starting values in a direct synthesis effort to find the parameters for improved performance.

17. J. McMunn and G. Jorgensen, "A Review of the Literature on Optimization Techniques and Minimax Structural Response Problems," Univ. of Minn., Inst. of Tech., Dept. of Aeronautics and Engineering Mech., Report TR 65-5 (Oct. 1966).

    This report performs the task indicated by the title using only a portion of the available literature. The min-max structural response problems include optimization of damping in mechanical systems and the time-optimal and near-optimal design of shock isolation devices. (See listing in next section of bibliography.)

18. Iu. I. Neimark, "Calculating An Optimal Vibration Isolator," *Mekh. Tverd. Tela.*, p. 182 (Sept-Oct. 1966) (in Russian).

    This paper presents an anlaytical study of a single-degree-of-freedom shock isolation system subject to a known input. An expression is derived to indicate the minimum rattlespace for which an isolator can be found that satisfies specified upper and lower bounds over a time interval.

19. W. D. Pilkey, E. H. Fey, J. F. Costello, T. Liber, and A. Kalinowski, "Shock Isolation Synthesis Study," SAMSO TR-68-388, Vol. I and II (1968), Space and Missile Systems Organization, USAF. San Bernardino, Calif.

    This report describes various techniques for what is termed in the monograph the *indirect method of design-parameter synthesis.* Listing and documentation are provided for a computer code which performs the time-optimal synthesis of single-degree-of-freedom systems with rattlespace and peak acceleration criteria. An LP preprocessing code for arranging the input in LP form for the multiple-isolator, multiple-degree-of-freedom shock isolation system with linear structural elements, and

very general forms of criteria, following the method of Appendix B, also are included. A study, including a dynamic programming code, is made for the extremal disturbance analysis of single-degree-of-freedom systems as described in Chapter 5 of the monograph.

20. D. M. Rogers, G. Urmston, and Ching-U Ip, "A Method for Designing Linear and Nonlinear Shock Isolation Systems for Underground Missile Facilities," BDS-TR-67-173, Ballistic Systems Division, USAF, San Bernardino, Calif. (June 1967).

    This report contains equations of motion for a rectangular package supported vertically by springs and isolated horizontally by nonlinear devices such as foam. Excitations are prescribed displacements at the springs and isolators. A factorial search procedure, including a computer code, to select optimum design parameters on the basis of minimum peak acceleration and displacement response is described. The algorithm amounts to attempting all combinations of parameters with a termination of the integration of equations as soon as any set of parameters leads to a violation of the constraints.

21. J. E. Ruzicka, "Characteristics of Mechanical Vibration and Shock," *Sound and Vibration*, 1 (4), 14-21 (1967).

    Sources, physical characteristics, and mathematical representations of typical shock and vibration environments are thoroughly discussed. Both deterministic (shock and harmonic vibration) and probabilistic (random vibration) disturbances are considered.

22. J. E. Ruzicka, "Passive and Active Shock Isolation," paper presented at NASA Symp. on Transient Loads and Response of Space Vehicles, Langley, Va., Nov. 1967.

    Available active isolation devices are surveyed and contemporary active isolation technology is compared with passive shock isolation techniques. Optimum synthesis and problems encountered in isolation design when both shock and vibration disturbances are anticipated are discussed briefly.

23. J. Ruzicka, "Active Vibration and Shock Isolation," SAE Paper 680747 (1968).

    This is an updated version of "Passive and Active Shock Isolation" (Ruzicka) and includes a discussion of contemporary hardware systems.

24. L. A. Schmit and R. L. Fox, "Synthesis of a Simple Shock Isolator," NASA CR-55 (June 1964).

    This report describes a direct optimum synthesis of an SDF system with a parallel linear spring and dashpot isolator; performance criteria are peak acceleration and rattlespace. The formulation accepts multiple, precisely defined disturbances. The optimum design parameters are found such that the maximum performance index for the class of disturbances is minimized while the constraints are not violated for any

input belonging to the class. This is a worst-disturbance direct optimum synthesis problem for multiple inputs as described in Chapter 6 of the monograph.

25. L. A. Schmit and E. F. Rybicki, "Simple Shock Isolator Synthesis with Bilinear Stiffness and Variable Damping," NASA CR-64710 (June 1965).
The "Synthesis of a Single Shock Isolator" work (Schmit-Fox) is extended to an isolator which has nine design parameters and is composed of a bilinear spring and time-dependent dashpot.

26. E. Sevin and W. D. Pilkey, "Min-Max Response Problems of Dynamic Systems and Computational Solution Techniques," *Shock and Vibration Bull. No. 36*, Pt. 5, 69-76 (Jan. 1967).
Extremal analysis and optimum synthesis problems for dynamic systems are considered. Mathematical programming formulations for time-optimal synthesis of shock isolation systems with fully prescribed inputs and for the worst disturbance of a given class of disturbances, which are considered in the monograph, are included.

27. E. Sevin and W. Pilkey, "Optimization of Shock Isolation Systems," Society of Automotive Engineers, Proceedings of the 1968 Aeronautic and Space Engineering and Manufacturing Meeting (1968).
A mathematical statement of the problem of optimum design of multi-isolator, multidegree-of-freedom shock isolation systems is presented, as is a general discussion of direct and indirect optimum synthesis including computational implementation.

28. E. Sevin, W. Pilkey and A. Kalinowski, "Computer-Aided Design of Optimum Isolation Systems," *Shock and Vibration Bull. No. 39*, Pt. 4, 1-13 (1969).
The same problem as "Optimization of Shock Isolation System" (Sevin-Pilkey) is treated, with concentration on the system identification phase of the indirect synthesis method.

29. V. A. Troitskii, "On the Synthesis of Optimal Shock Isolators," *J. Appl. Math. Mech.* (Translation of *Prikl. Mat. Mekh.*) **31** (4), 649-654 (1967), Pergamon Press.
The min-max SDF optimum isolator problem formulated in this monograph is posed. Both the optimum isolator configuration and design parameters are sought. Solutions are obtained using classical variational approaches after the maximum-deviation performance index is replaced by a quadratic form.

30. J. Wolkovitch, "Techniques for Optimizing the Response of Mechanical Shock and Vibration," SAE Paper 680748 (1968).
This is a control-theory-oriented survey of some available optimization techniques for shock and vibration isolation systems. Various optimization criteria, including maximum deviation in time and time-integral forms, are discussed and several single-degree-of-freedom problems are solved in detail.

HARMONIC VIBRATION ISOLATION SYSTEMS

31. J. McMunn and G. Jorgensen, "A Review of the Literature on Optimization Techniques and Minimax Structural Response Problems," University of Minnesota, Institute of Technology, Department of Aeronautics and Engineering Mechanics, TR-65-5, October 1966.

The "minimax" problems considered in this review paper deal mostly with the selection of optimum damping rates for unconstrained mechanical systems under harmonic excitation. Some consideration is given to the time-optimal and near-optimal design of shock isolation systems. The first category of literature dates prior to computer-oriented methods and is fairly complete. The following entries marked by an asterisk are taken from this review. More recent extensions of analytical approaches to optimum damping have not been reviewed in connection with the monograph.

32. *F. R. Arnold, "Steady-State Behavior of Systems Provided with Non-Linear Dynamic Vibration Absorbers," *J. of Appl. Mech.,* **22**, 487-492 (1955).

The response of vibrating systems subjected to sinusoidal excitations and to the action of nonlinear dynamic vibration absorbers is determined by the Ritz method. One of the most striking characteristics of system response is the apparent existence of up to three modes of oscillation for a single value of disturbance frequency.

33. *D. B. Bogy and P. R. Paslay, "Evaluation of Fixed Point Method of Vibration Analysis for Particular System with Initial Damping," *J. Eng. Ind.,* **85** (3), 233-236 (1963).

The maximum steady-state response of a linear damped two-degree-of-freedom system is minimized by determining the optimum damping constant for an additional single damper. This is accomplished by both a well-known approximate method (fixed-point method) and an exact numerical method. Since the approximate method does not take into account the initial damping in the system, attention is directed toward determining the influence of initial damping on the optimum value for the single damper.

34. *J. E. Brock, "A Note on the Damped Vibration Absorber," *J. Appl. Mech.,* **13**, A284 (1946).

Formulas for optimum damping for three cases of the dynamic vibration absorber with damping are presented, along with the method of derivation of each. The three cases are (a) optimum tuning, (b) constant tuning, and (c) Lanchester type damper (viscous damping).

35. *J. E. Brock, "Theory of the Damped Dynamic Vibration Absorber for Inertial Disturbances," *J. Appl. Mech.,* **16**, A86 (1949).

This paper deals with a vibration absorber for a system having the driving force proportional to the square of the driving frequency.

The criterion for optimum tuning is determined following an analysis similar to that given by Den Hartog for the usual dynamic absorbers.

36. *W. J. Carter and F. C. Liu, "Steady-State Behavior of Non-Linear Dynamic Vibration Absorber," *J. Appl. Mech.*, 28 (1), 67-70 (1961).

The Frahm-type dynamic vibration absorber is analyzed for the case where both main spring and absorber spring have nonlinearities of the Duffing type. A one-term approximate solution is assumed for the motion of the two masses, and the resulting amplitude equation is solved using a graphical procedure. An optimum dynamic vibration-absorber system for variable-frequency excitations consists of a hardening main spring with a softening absorber spring.

37. *E. Hahnkamm, "Die Dampfung von Fundamentschwingungen bei Veranderlicher Erreger Frequenz," *Ing. Arch.*, 4 (1933).

In this early work, Hannkamm found the changes in the amplitudes of each of the two maxima of the unit vibration response of a two-degree-of-freedom linear system as the damping coefficient of the single linear dashpot is changed.

38. *A. R. Henney, "Damping of Continuous Systems," *Engineer,* 215 (5529), 572-574 (1963).

It is shown that for some simple continuous systems (beams damped at one point and harmonically forced at other points), theory and experiment agree well for the choice of concentrated damping which will optimize the response over a given frequency range. The sensitivity of maximum response to variation of damping is approximated by considering that, as damping tends to large or small values, the maximum response tends to an infinite resonance and behavior of the beam may be approximated by a single-degree-of-freedom system vibrating near a resonance mode.

39. *A. Henney and J. P. Raney, "The Optimization of Damping of Four Configurations of a Vibrating Beam," *J. Eng. Ind.*, 85 (3) 259-264 (1963).

This paper contains a development of approximate analytic expressions for optimum damping of a uniform free-free beam connected to the support by one or two viscous dampers and excited at different points. The configurations investigated are found to be relatively insensitive to deviations of the damping form optimum.

40. *E. M. Kerwin, Jr., "Damping of Flexural Waves by a Constrained Viscoelastic Layer," *J. Acoust. Soc. Amer.*, 31 (7), 952 (1959).

This paper presents a quantitative analysis of the damping effectiveness of a constrained viscoelastic layer. The damping factors determined experimentally agree well with those calculated theoretically. The theoretical expressions for the damping effectiveness are based on the mechanism of shear energy-loss.

41. *F. M. Lewis, "Extended Theory of Viscous Vibration Damper," *J. Appl. Mech.*, 22, 377 (1955).

This paper extends the theory of the viscous vibration damper, either tuned or untuned, to multimass torsional systems and shows how an optimum damper can be designed for any installation. This extended-damper theory is based on the fixed-point theorem.

42. T. Liber, "Optimum Shock Isolator Synthesis," AFWL-TR-65-82 (July 1966).

This report contains an appendix concerned with the optimum performance characteristics for linear (spring, dashpot) single-degree-of-freedom systems under harmonic input. The curves, which are presented for several ranges of values of material parameters, are appropriate for the simple isolator or absorber (fixed base) systems.

43. J. C. McMunn, "Multi-Parameter Optimum Damping in Linear Dynamical Systems," unpublished doctoral dissertation, University of Minnesota 1967.

The problem of determining optimum damping rates of large mechanical systems with multiple dampers and harmonic inputs is considered. Damping is defined to be optimum if a peak displacement response is minimized for an input frequency interval. Detailed consideration is given to a multiple-degree-of-freedom linear system for which the optimum damping is found by direct synthesis with a worst-disturbance analysis applied at each iteration. No response constraints are involved. Two multiple-degree-of-freedom, multiple-damper discrete systems and a column with distributed complex modulus damping are studied as example problems. The literature survey by McMunn and Jorgensen in item 31 is summarized.

44. J. C. McMunn and R. Plunkett, "Multi-Parameter Optimum Damping in Linear Dynamical Systems," ASME Vibrations Conference Paper 69-VIBR-42.

This paper is a summarization of McMunn's doctoral dissertation of the same title. item 43.

45. *T. J. Mentel, "Visco-Elastic Boundary Damping of Beams and Plates," *J. Appl. Mech.*, 31 (1), 61-71 (1964).

This paper presents experiments on boundary-damped beams that identify the effectiveness of axial and transverse motions in producing energy dissipation. Experiments that test the damping effectiveness of small insets of viscoelastic adhesive are described.

46. *V. H. Neubert, "Dynamic Absorbers Applied to Bar that has Solid Damping," *J. Acoust. Soc. Amer.*, **36** (4), 673 (1964).
The theoretical steady-state response of an axially excited bar with solid damping is determined. The effect of adding one or two dynamic absorbers is considered, and the optimization of the absorber damping is discussed for constant damping in the bar.

47. *B. E. O'Connor, "The Viscous Torsional Vibration Damper," *SAE Trans.* **1**, 87-97 (1947).
This article points out the drawbacks of the untuned damper with dry friction. There is a development of the theory of application of the untuned damper with viscous damping.

48. *J. Ormondroyd and J. P. Den Hartog, "Theory of the Dynamic Vibration Absorber," *Trans. ASME,* **50** (1928), ATM-50-7.
In this classical paper, it is first shown that a vibration absorber without damping completely annihilates the vibration at its own frequency, but creates two critical speeds in the machine system. Therefore, it is suitable only for constant-speed machinery. With damping, the absorber can diminish the vibration of a machine of variable speed. The analysis of its operation in simple cases is presented.

49. I. L. Paul and E. K. Bender, "Active Vibration Isolation and Active Vehicle Suspension," MIT Dept. of Mech. Engr. (Nov. 1966) (PB 173,648).
The limiting performance characteristics for a generic single-degree-of-freedom system and the optimum spring-dashpot system subject to harmonic inputs are discussed. These curves are based on rattlespace and peak acceleration criteria although they differ somewhat in form from those given in the monograph. Most of this report is concerned with isolation systems for random disturbances.

50. *R. Plunkett, "The Calculation of Optimum Concentrated Damping for Continuous Systems," *J. Appl. Mech.*, **25** (2), 219-224 (1958).
The approach is a generalization of that employed by Lewis, Den Hartog, and Ormondroyd. Four specific problems are considered. Some general conclusions are that the vibration velocity and vibratory force are not necessarily in phase at maximum amplitude for optimum damping and that the decay rate at optimum damping is not necessarily related to the amplification at resonance.

51. *R. Plunkett, "Vibration Response of Linear Damped Complex Systems," *J. Appl. Mech.*, **30** (1), 70-74 (1963).
This paper develops two approximate expressions for the change in all of the response maxima of a multidegree or continuous system as the

coefficient of the single linear damper is changed. One of these approximations is derived from a perturbation solution about the min-max values, and the other is derived from an expansion in normal modes. These expressions are useful in determining the sensitivity of the maximum response value to small changes in the damping coefficient.

52. *R. Plunkett and C. H. Wu, "Attenuation of Plane Waves in Semi-Infinite Composite Bar," *J. Acoust. Soc. Amer.*, 37 (1), 28-30 (1965).
It is shown that the maximum attenuation of the propagating waves occurs for an optimum value of loss tangent of the shear modulus. Wave numbers depend on complex shear modulus, frequency, and dimensions of the bar.

53. *R. E. Roberson, "Synthesis of a Non-Linear Dynamic Vibration Absorber," *J. Franklin Inst.*, **254**, 205-230 (1952).
A secondary system is attached to a linear undamped vibrating system with one degree of freedom by means of a nonlinear spring. It is desired to find optimum values of the coefficients of this spring such that the vibration amplitude of the primary system is kept below unity for as large a band of exciting frequencies as possible. The first approximation by the Duffing iteration method is used to obtain the response in terms of the system parameters. For the synthesis criterion used, the nonlinear absorber offers a significant advantage over the corresponding linear absorber.

54. *J. F. Springfield and J. P. Raney, "Experimental Investigation of Optimum End Supports for a Vibrating Beam," *Exp. Mech.*, 2 (12), 366-372 (1962).
The problem investigated is the extent to which the near-resonant response of the beam to a concentrated harmonic force could be limited in a predictable manner by applying vibration absorbers to the ends of the beam. The experimentally determined response amplitude, frequencies of fixed points, and optimum values of damping agree well with theory.

RANDOM VIBRATION ISOLATION SYSTEMS

55. E. K. Bender, "Optimum Linear Preview Control with Application to Vehicle Suspension," ASME Paper 67-WA/Aut-1.
A study of performance limits and direct optimum synthesis of linear isolation systems with sensors is summarized. The Wiener filter approach is used to establish the optimum transfer function. The resulting system is analyzed for response to a step pulse disturbance. Terrain environments are characterized in the same fashion as in "On the Optimization of Vehicle Suspensions Using Random Process Theory" (Bender, Karnopp, Paul).

56. E. K. Bender, "Optimization of the Random Vibration Characteristics of Vehicle Suspension," MIT Dept. of Mech. Eng., unpublished Sc.D. dissertation, June 1967.

This is the most comprehensive of the documents by the MIT group studying optimum suspension systems on the basis of the Wiener filter. Many derivations and explanations that are sketchily presented in other reports and papers are given full, detailed consideration here.

57. E. K. Bender and I. L. Paul, "Analysis of Optimum and Preview Control of Active Vehicle Suspension," MIT Dept. of Mech. Eng. Rpt DSR-76109-6, U.S. Dept. of Transportation (Sept. 1967) (Clearinghouse No. PB 176137).

This is an interim report on the research described in "Optimization of the Random Vibration Characteristics of Vehicle Suspension" (Bender).

58. E. K. Bender, "Optimum Linear Control of Random Vibrations," *Proc. 8th Joint Automat. Contr. Conf.*, June 28-30, 1967.

This is a preliminary version of the paper "On the Optimization of Vehicle Suspensions Using Random Process Theory" (Bender, Karnopp, Paul).

59. E. K. Bender, D. C. Karnopp, and I. L. Paul, "On the Optimization of Vehicle Suspensions Using Random Process Theory," ASME Paper No. 67-TRAN-12, *Mech. Eng.*, 89, 69 (1967).

This review of work performed on vehicle suspension systems is the most complete open-literature source on the approach pursued in Chapter 8 of the monograph: Indeed, much of the material was drawn directly from this paper. A design chart useful in selecting optimum parameters for a spring-dashpot isolator of a flexible base system is given along with a detailed numerical example of an optimum design problem.

60. E. K. Bender, "Some Fundamental Limitations of Active and Passive Vehicle-Suspension Systems," SAE Paper 680750 (1968).

This paper contains a summary of the work presented in "On the Optimization of Vehicle Suspensions Using Random Process Theory" (Bender, Karnopp, Paul) and "Optimum Linear Preview Control with Application to Vehicle Suspension" (Bender), and also includes a brief discussion of suspension system characteristics which are desirable in reducing lateral acceleration during rolling motion of a ground vehicle.

61. T. F. Derby and P. C. Calcaterra, "Response and Optimization of an Isolation System with Relaxation Type Damping," *Shock and Vibration Bulletin No. 40* (1970).

The authors consider relaxation-type damping to be an isolator element composed of either a Voigt viscoelastic model in series with an elastic spring, or a standard linear solid viscoelastic model. Inputs are impulse and white noise acceleration of the base. An analytical direct optimal synthesis study is performed of a single-degree-of-freedom system on the

basis of the type of acceleration and rattlespace criteria formulated in Chapters 5, 6, and 7 of the monograph. Peak acceleration-vs-rattlespace tradeoffs for elements with optimum parameters are plotted as dimensionless design curves and compared with the limiting performance characteristics. This is a thorough study of the problem posed.

62. D. C. Karnopp, "Applications of Random Process Theory to the Design and Testing of Ground Vehicles," *Transp. Res.*, 2, 269-278 (1968).
    The initial sections of this paper contain an interesting, fundamental discussion of the statistical characterization of ground terrain. This is the spectral density characterization used to advantage for the optimum isolation system design studies of Chapter 8 of the monograph.

63. D. C. Karnopp, "Continuum Model Study of Preview Effects in Actively Suspended Long Trains," *J. Franklin Inst.*, 285 (4), 251-260 (1968).
    The paper is an initial effort at showing that in a long train the cars themselves can be used as sensors for the type of preview suspension system discussed in Bender's "Optimum Linear Preview Control with Application to Vehicle Suspension."

64. D. C. Karnopp and A. K. Trikha, "Comparative Study of Optimization Techniques for Shock and Vibration Isolation," AFOSR 68-0242 (Jan. 1968) and *J. Eng. Ind.*, 91 (4), 1128-1132 (1969).
    Several optimum and near-optimum isolation systems are considered with respect to min-max, quadratic, and expected mean-square value criteria. It is shown that the systems designed on the basis of one criterion do not necessarily respond favorably with respect to other criteria. The report version contains several important appendixes not included in the paper, Ref. 1.

65. G. C. Newton, L. A. Gould, and J. F. Kaiser, *Analytical Design of Linear Feedback Controls*, John Wiley and Sons, New York, 1957.
    Most of the work surveyed in Chapter 8 of this monograph represents an isolation-system-oriented version of the material on linear feedback controls presented in the book. This includes the concept of initiating an optimum design by determining, on the basis of the problem specifications, certain characteristics of the absolute optimum linear system. This book can be used as a source of thorough and rigorous de[illegible]ons of certain brief discussions in Chapter 8 of the monograph.

66. I. L. Paul and E. K. Bender, "Partial Bibliography on Subjects Related to Active Vibration Isolation and Active Vehicle Suspensions," MIT, Dept. of Mech. Eng. Projects Laboratory Rpt DSR-76109-2, Clearinghouse No. PB 173649 (Nov. 1966).
    This report contains a listing of some of the available literature related to vibration isolation including random input characterization and optimum design. All entries are classified according to subject; entries are not annotated.

67. I. L. Paul and E. K. Bender, "Active Vibration Isolation and Active Vehicle Suspension," MIT, Dept. of Mech. Eng. Rpt. DSR-76109-1, Clearinghouse No. PB 173648 (Nov. 1966).

This is a preliminary report of some of the work described in "Optimization of the Random Vibration Characteristics of Vehicle Suspension" (Bender).

68. A. Seireg and L. Howard, "An Approximate Normal Mode Method for Damped Lumped Parameter Systems," *J. Eng. Ind.*, 89 (4), 597-604 (1967).

A computational search routine is used to select optimum design parameters (frequency and damping ratios) for a simple damped absorber consisting of a mass connected by a linear spring to a rigid base on one side and to another mass by a spring and dashpot in parallel in the other direction (i.e., the flexible models of Examples 5 and 6). The first mass is subject to white-noise random excitation. The parameter ratios are plotted as functions of the mass ratio and compared to the curves presented in Den Hartog's *Mechanical Vibrations* for sinusoidal loading.

69. A. R. Trikha and D. C. Karnopp, "A New Criterion for Optimizing Linear Vibration Isolator Systems Subject to Random Input," *J. Eng. Ind.*, **91** (4), 1005-1010 (1969).

The proposed criterion deals with values of displacement and acceleration for which the probability of exceeding is less than a desired value. The random input must be stationary and gaussian. The problem is reduced to a version of the Wiener filter synthesis solution discussed in Chapter 8 of the monograph.

70. J. Wolkovitch, "Techniques for Optimizing the Response of Mechanical Systems to Shock and Vibration," SAE Paper 680748 (1968).

This is an interesting survey of some of the literature and techniques available for the optimization of isolation systems subject to shock and vibration as seen by a control engineer. Emphasis is placed on analytical techniques suitable for simple systems. Discussions on criteria include an evaluation of an integral representation of a min-max performance index and a warning of possible pitfalls of including a constraint in a performance index.

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